Lossy Source Coding for a Cascade Communication System with Side-Informations

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***Abstract*— We investigate source coding in a cascade communication system consisting of an encoder, a relay and an end terminal, where both the relay and the end terminal wish to reconstruct source X with certain fidelities. Additionally, side-informations Z and Y are available at the relay and the end terminal, respectively. The side-information Z at the relay is a physically degraded version of side-information Y at the end terminal. Inner and outer bounds for the rate distortion region are provided in this work for general discrete memoryless sources. The rate distortion region is characterized when the source and side-informations are jointly Gaussian and physically degraded. The doubly symmetric binary source is also investigated and the inner and outer bounds are shown to coincide in certain distortion regimes. A complete equivalence of the rate-distortion region is established between the problem being considered and the side-information scalable source coding problem, when there is no side-information at the relay. As a byproduct, the same equivalence can be established between the well-known successive refinement problem and Yamamoto's cascade communication system, without relying on their rate-distortion characterization.**

I. Introduction

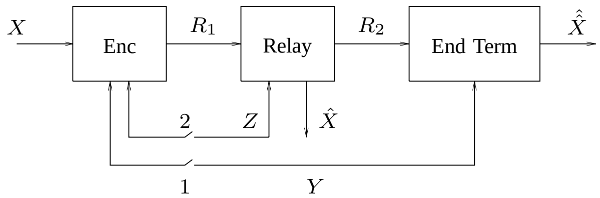

Fig. 1. The three terminal serial network with encoder side informations

We consider source coding for the cascade communication system with side-informations (CC-SI) depicted in Fig. 1. In this system, the encoder observes a discrete memoryless source (DMS) X, and wants to communicate it within fidelities of D_1 and D_2, respectively, to the relay and the end terminal. The link of rate R_1 between the encoder and the relay, and that of rate R_2 between the relay and the end terminal, are noiseless. The relay and the end terminal have access to discrete memoryless side-informations Z and Y, respectively, which are physically degraded as $X \leftrightarrow Y \leftrightarrow Z$, i.e., $p(x,y,z) = p(x)p(y|x)p(z|y)$.

This system models communication in a network where the observer (the encoder) cannot communicate to all the recipient users (decoders) due to certain constraints, such as power consumption. Therefore, some of the users in the network have to act as relays for the decoders not directly linked to the observer. Additionally, the relay and the decoders can have access to side-informations. We focus on the special case where the side-informations are physically degraded with the relay having access to the side-information of poorer quality. This particular choice is motivated by sensor networks where the end terminal is a data collection center and therefore has access to greater amount of side-information as compared to the relays. Though our emphasis is on the case when the encoder has no access to any side information, we also provide bounds for the case when the encoder has access to the decoder side informations.

The cascade communication system without side-information was considered by Yamamoto in [1], who characterized the complete rate-distortion region. More recently an inner bound was found for a related problem in [2], where the encoder observes two correlated sources, and the relay and the end terminal are interested in the individual sources losslessly, with only the end terminal having access to side-information; it will be shown that the inner-bound provided here improves upon that given in [2].

In this work, we first provide inner and outer bounds for the rate-distortion region of the CC-SI problem. The rate distortion region is completely characterized when the source and side-information are jointly Gaussian and physically degraded. The doubly symmetric binary source is investigated and the inner and outer bounds are shown to coincide in certain distortion regimes.

Though we are not able to characterize the complete rate-distortion region, we make a connection between the CC-SI problem and the side-information scalable (SI-scalable) source coding problem considered in [8]. In the SI-scalable problem, a common message is given to two decoders, who have degraded side-informations; furthermore, a private message is given to the decoder with the poorer quality side-information. For the special case where the poor quality side-information does not exist, we establish an equivalence between the rate distortion regions of the SI-scalable problem and the CC-SI problem. It is straightforward to generalize this equivalence to multiple stages when the intermediate relays do not have access to side-informations. As a special case, it implies that the solution for Yamamoto's cascade communication system [1] immediately provides the solution for the well-known successive refinement problem [4], and vice versa.

The rest of the paper is organized as follows. In Section II we introduce notations and necessary backgrounds. Inner and outer bounds are provided in Section III and IV. Section V discusses the equivalence between the CC-SI and the SI-scalable problem. The Gaussian source and the doubly symmetric binary source are treated in Section VI and X, respectively. Section XI concludes the paper.

II. Notation and Preliminaries

We use capital letters to describe random variables and the corresponding small letters to describe their realizations. A discrete memoryless source is an infinite sequence $\{X_i\}_1^\infty$ of independent copies of a random variable X with discrete support $\mathcal{X}$ and a generic distribution $p(x)$. Let (Y, Z) be memoryless random variables physically degraded with respect to X as $X \leftrightarrow Y \leftrightarrow Z$ and taking values in finite alphabet, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Let $\hat{\mathcal{X}}$ be the reconstruction alphabet of both decoders and let $d : \mathcal{X} \times \hat{\mathcal{X}} \to [0, \infty)$ be a distortion measure which acts on vector pairs $(x^n, \hat{x}^n)$ as $d(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^n d(x_i, \hat{x}_i)$.

Definition 1: An (n, M_1, M_2, D_1, D_2) code consists of two encoding functions ϕ_1 and ϕ_2,

$$\phi_1 \;:\; \mathcal{X}^n \to I_{M_1}, \quad \phi_2 \;:\; I_{M_1} \times \mathcal{Z}^n \to I_{M_2}$$

and two decoding functions [1] ψ_1 and ψ_2,

$$\psi_1 \;:\; I_{M_1} \times \mathcal{Z}^n \to \hat{\mathcal{X}}^n, \quad \psi_2 \;:\; I_{M_2} \times \mathcal{Y}^n \to \hat{\mathcal{X}}^n$$

where $I_k = \{1, \dots, k\}$ such that

$$\mathbb{E}d(X^n, \psi_1(\phi_1(X^n), Z^n)) \le D_1,$$
$$\mathbb{E}d(X^n, \psi_2(\phi_2(\phi_1(X^n), Z^n), Y^n)) \le D_2.$$

Definition 2: A rate pair (R_1, R_2) is said to be (D_1, D_2)-achievable, if for any $\epsilon > 0$ and sufficiently large n, there exists an $(n, M_1, M_2, D_1+\epsilon, D_2+\epsilon)$ code such that $\frac{1}{n}\log(M_1) \le R_1+\epsilon$ and $\frac{1}{n}\log(M_2) \le R_2+\epsilon$.

We denote the collection of all (D_1, D_2) achievable rate pairs (R_1, R_2) as $\mathcal{R}(D_1, D_2)$, and this is the region to be characterized.

III. Inner Bound

In this section we give inner bounds for $\mathcal{R}(D_1, D_2)$.

A. Inner Bound - Both Switches Open

Define $\mathcal{R}_i(D_1, D_2)$ to be the set of all rate pairs (R_1, R_2) for which there exist random variables (U, V, W) with finite alphabets, and jointly distributed with (X, Y, Z) as $p(x, y, z, u, v, w) = p(z)p(y|z)p(x|y)p(u, v, w|x)$ (i.e., $(W, V, U) \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$)) which satisfy the following conditions

$$R_1 \ge I(X; U|Z) + I(X; W|U, Z) + I(X; V|U, Y) \tag{1}$$

$$R_2 \ge I(X; U, V|Y) \tag{2}$$

and for which there exist mappings $f(U, W, Z), g(U, V, Y)$ to the reconstruction alphabet $\hat{\mathcal{X}}$ such that

$$\mathbb{E}[d(X, f(U, W, Z))] \le D_1, \;\; \mathbb{E}[d(X, g(U, V, Y))] \le D_2. \tag{3}$$

The following theorem provides an inner bound to the rate distortion region.

***Theorem** 1:*

$$\mathcal{R}_i(D_1, D_2) \subseteq \mathcal{R}(D_1, D_2).$$

Outline of proof:
Here we outline the proof by providing a random coding scheme. Let ϵ and ϵ_i, $i = \{1, \dots, 7\}$ be some small positive quantities. A codebook of size $2^{n[I(X;U)+\epsilon_1]}$ is generated by independently sampling from the typical set $T_\epsilon(U^n)$; label them as $u^n(i)$ for $i = 1, \dots, 2^{n[I(X;U)+\epsilon_1]}$. For each $u^n(i)$, generate a codebook of size $2^{n[I(X;V|U)+\epsilon_2]}$ by independently

[1] For simplicity, we assume the same reconstruction alphabet and distortion measure are used at both the relay and the end terminal.

sampling from the typical set $T_\epsilon(V^n|u^n(i))$. Similarly, for each $u^n(i)$, generate a codebook of size $2^{n[I(X;W|U)+\epsilon_3]}$ by independently sampling from the typical set $T_\epsilon(W^n|u^n(i))$. The codewords in the U, V and W codebooks are binned uniformly at random into $2^{n[I(X;U|Z)+\epsilon_4]}$, $2^{n[I(X;V|U,Y)+\epsilon_5]}$ and $2^{n[I(X;W|U,Z)+\epsilon_6]}$ bins, respectively; these bins will be referred to as the encoder-bins. The codewords and the bins are revealed to both the relay and the end terminal. Additionally, the codewords in the U codebook are again binned uniformly at random into $2^{n[I(X;U|Y)+\epsilon_7]}$ bins and the binning is revealed to both the relay and the end terminal; this will be referred to as the relay-bin.

The encoder observes x^n and finds codewords $\hat{u}^n, \hat{v}^n, \hat{w}^n$ such that $(x^n, \hat{u}^n, \hat{v}^n)$ and $(x^n, \hat{u}^n, \hat{w}^n)$, respectively, are jointly typical. The encoder then finds the encoder-bin indices corresponding to the three codewords and sends them to the relay. The decoder at the relay uses side-information z^n to find the pair $(\hat{u^n}, \hat{w^n})$ from the indicated bins. The encoder at the relay then finds the relay-bin index for $\hat{u}^n$ and sends it, along with the encoder-bin index of v^n, to the end terminal. The end terminal uses the side-information y^n to find the pair $(\hat{u}^n, \hat{v}^n)$ in the indicated bins. By choosing the rate and appropriate $\epsilon_1, ..., \epsilon_7$, it can be shown using conventional techniques that the probability of encoding/decoding failure can be made arbitrarily small and the desired distortions can be achieved. $\square$

Intuitively, U^n can be seen as a common message sent to both the relay and the end terminal, while V^n, W^n are two private messages, respectively, for the end terminal and the relay.

Remark: The coding scheme given above can be straightforwardly applied to the problem considered in [2], where the multisource (S_1, S_2) is observed at the encoder, the relay is interested in lossless reconstruction of S_1, and the end terminal is interested in lossless reconstruction of S_2; side-information Y is available at the end terminal and the relay has no access to any side-information. Specializing our coding results gives the following achievable region for some random variable U such that $U \leftrightarrow (S_1, S_2) \leftrightarrow Y$:

$$R_1 \geq I(S_1 S_2; U) + H(S_1|U) + H(S_2|UY) \quad (4)$$

$$R_2 \geq I(S_1 S_2; U|Y) + H(S_2|UY). \quad (5)$$

Comparing with the inner-bound given in [2], while the expression for rate R_1 is essentially the same, the rate R_2 is improved by an amount of $I(U;Y)$.

We also present the inner bounds for the case when the encoder has access to partial side informations. In particular, three cases corresponding to three different switch configurations in Fig. 1 are examined. The inner bounds follow immediately from the inner bound for the case with no encoder side information via the "super-source" argument, where, the source stream and the side information streams available at the encoder are treated as a unified super-source.

B. Inner Bound - Both Switches Closed

The inner bound $\mathcal{R}_i(D_1, D_2)$ is the set of all rate pairs (R_1, R_2) for which there exist random variables (U, V, W) with finite alphabets, and jointly distributed with (X, Y, Z) as $p(x, y, z, u, v, w) = p(z)p(y|z)p(x|y)p(u|x, y, z)p(v|u, x, y)p(w|u, x, z)$ which satisfy the following conditions

$$R_1 \geq I(XY; U|Z) + I(X; W|U, Z) + I(X; V|U, Y) \quad (6)$$

$$R_2 \geq I(X, Z; U, V|Y), \quad (7)$$

and for which there exist mappings $f(U, W, Z), g(U, V, Y)$ to the reconstruction alphabet $\hat{\mathcal{X}}$ such that

$$\mathbb{E}[d(X, f(U, W, Z))] \leq D_1,\ \mathbb{E}[d(X, g(U, V, Y))] \leq D_2 \quad (8)$$

C. Inner Bound - Switch 1 closed, Switch 2 open

The inner bound $\mathcal{R}_i(D_1, D_2)$ is the set of all rate pairs (R_1, R_2) for which there exist random variables (U, V, W) with finite alphabets, and jointly distributed with (X, Y, Z) as $p(x, y, z, u, v, w) = p(z)p(y|z)p(x|y)p(u|x, y)p(v|u, x, y)p(w|u, x)$ which satisfy the following conditions

$$R_1 \geq I(XY; U|Z) + I(X; W|U, Z) + I(X; V|U, Y) \quad (9)$$

$$R_2 \geq I(X; U, V|Y) \quad (10)$$

and for which there exist mappings $f(U, W, Z), g(U, V, Y)$ to the reconstruction alphabet $\hat{\mathcal{X}}$ such that

$$\mathbb{E}[d(X, f(U, W, Z))] \leq D_1,\ \mathbb{E}[d(X, g(U, V, Y))] \leq D_2. \quad (11)$$

D. Inner Bound - Switch 1 open, Switch 2 closed

The inner bound $\mathcal{R}_i(D_1, D_2)$ is the set of all rate pairs (R_1, R_2) for which there exist random variables (U, V, W) with finite alphabets, and jointly distributed with (X, Y, Z) as $p(x, y, z, u, v, w) =$

$p(z)p(y|z)p(x|y)p(u|x,z)p(v|u,x)p(w|u,x,z)$ which satisfy the following conditions

$$R_1 \geq I(X;U|Z) + I(X;W|U,Z) + I(X;V|U,Y) \quad (12)$$

$$R_2 \geq I(X,Z;U,V|Y), \quad (13)$$

and for which there exist mappings $f(U,W,Z), g(U,V,Y)$ to the reconstruction alphabet $\hat{\mathcal{X}}$ such that

$$\mathbb{E}[d(X,f(U,W,Z))] \leq D_1,\ \mathbb{E}[d(X,g(U,V,Y))] \leq D_2. \quad (14)$$

IV. Outer Bound

Heegard and Berger [7] considered the rate-distortion problem when there are (in general, more than) two decoders with degraded side-informations, and the encoder is required to send a common message to both the decoders for lossy reconstruction. The rate-distortion function for this problem is denoted here as $R_{HB}(D_1,D_2)$, where D_1, respectively D_2, is the distortion at the decoder with poorer quality side-information Z, respectively that with better quality side-information Y. It is given by

$$R_{HB}(D_1,D_2) = \min_{P(D_1,D_2)} [I(X;U|Z) + I(X;V|U,Y)] \quad (15)$$

where $P(D_1,D_2)$ is the set of all random variables (U,V) jointly distributed with (X,Y,Z) such that the following conditions are satisfied (i) $Z \leftrightarrow Y \leftrightarrow X \leftrightarrow (U,V)$ is a Markov string, (ii) There exist deterministic functions $f(U,Z)$ and $g(U,V,Y)$ such that

$$\mathbb{E}[d(X,f(U,Z))] \leq D_1,\ E[d(X,g(U,V,Y))] \leq D_2. \quad (16)$$

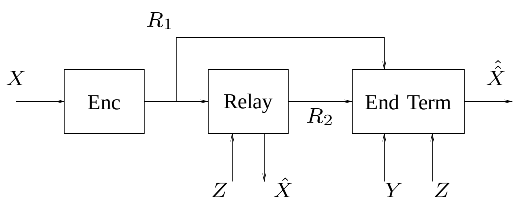

Fig. 2. Lower bound for R_1

Denote the well-known Wyner-Ziv rate distortion function with source X and side-information Y as $R_{WZ}(D)$. If additional side-information Z is also available at the encoder, then consider the setup with the super-source (X,Z) observed at the encoder and side-information Y observed at the decoder, and with the distortion measure depending on only X. Clearly this still falls into the framework of Wyner-Ziv coding. Denote the corresponding rate distortion function as $R^*_{WZ}(D)$.

Now define $\mathcal{R}_o(D_1,D_2)$ as follows

$$\mathcal{R}_o(D_1,D_2) = \big\{(R_1,R_2) : R_1 \geq R_{HB}(D_1,D_2),\ R_2 \geq R^*_{WZ}(D_2)\big\}. \quad (17)$$

The following theorem provides an outer bound for the rate-distortion region.

***Theorem** 2:*

$$\mathcal{R}(D_1,D_2) \subseteq \mathcal{R}_o(D_1,D_2).$$

For a system with common encoder and decoder side-information Y, and additional encoder side-information Z, it is known that Z does not help, and hence the rate distortion function is just the conditional form $R_{X|Y}(D)$. On the other hand, this system clearly provides a lower bound for $R^*_{WZ}(D)$. Therefore we have the following corollary.

Corollary

$$\mathcal{R}(D_1,D_2) \subseteq \big\{(R_1,R_2) : R_1 \geq R_{HB}(D_1,D_2),\ R_2 \geq R_{X|Y}(D_2)\big\}. \quad (18)$$

Proof of Theorem 2: The achievable region can only expand if we let the terminal decoder have access to the message sent on the first link, as well as the side-information Z. In this new setting (see Fig. 2), the rate R_2 can be set to zero without effecting rate R_1. Indeed, for any valid code, the message sent on the second link is a function of the message on the first link and side-information Z^n, and as such it can be constructed at the end terminal directly. Removing the second link leads to the problem considered by Heegard and Berger, with the decoders having access to the degraded side-informations Z and (Y,Z), respectively. Denote the rate-distortion function for this problem as $R^*_{HB}(D_1,D_2)$. We have

$$R_1 \geq R^*_{HB}(D_1,D_2).$$

We now show that $R_{HB}(D_1,D_2) = R^*_{HB}(D_1,D_2)$. Clearly it is true that $R_{HB}(D_1,D_2) \geq R^*_{HB}(D_1,D_2)$. On the other hand, in the original Heegard-Berger problem whose rate-distortion region is characterised by (4), let the decoder with the better quality side-information Y generate a random variable $\hat{Z}$ according to $p(Z|Y)$. Let us denote the rate-distortion function for the Heegard-Berger problem with the degraded side-informations Z and $(Y,\hat{Z})$, respectively, as $R^{**}_{HB}(D_1, D_2)$. Clearly, since this forms a subset of possible strategies, we have $R_{HB}(D_1, D_2) \leq R^{**}_{HB}(D_1, D_2)$. It was pointed out in [7] that the rate-distortion function only depends on the marginals of the source and the individual side-informations. Since $X \leftrightarrow Y \leftrightarrow Z$ is a Markov string, the marginals $p(X, Y, \hat{Z})$ and $p(X, Y, Z)$ are identical, we have that $R^{**}_{HB}(D_1, D_2) = R^{*}_{HB}(D_1, D_2)$. This proves that $R_{HB}(D_1, D_2) \leq R^{*}_{HB}(D_1, D_2)$.

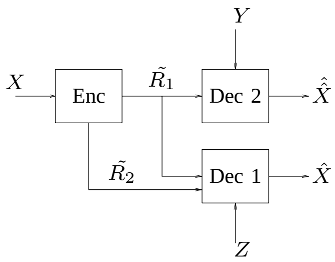

Fig. 3. SI-scalable problem

The rate R_2 could be lowered by letting the encoder and the relay cooperate. This leads to the problem setup with the super-source (X, Z) observed at the encoder and side-information Y observed at the decoder. It follows that $R_2 \geq R^{*}_{WZ}(D)$. □

We remark here that outer bounds can be obtained similarly for the case when the encoder has access to side informations.

V. Connection with the SI-scalable Problem

In this section we demonstrate the connection between the CC-SI problem and the SI-scalable problem considered in [8] (depicted in Fig. 3). The equivalence of the rate distortion region of the two problems is established for the case when Z is a constant. Define $\tilde{\mathcal{R}}(D_1, D_2)$ to be all the pairs $(\tilde{R}_1 + \tilde{R}_2, \tilde{R}_1)$, for which the rate pair $(\tilde{R}_1, \tilde{R}_2)$ is achievable in the SI-scalable problem. We have the following theorem.

***Theorem** 3:* If Z is a constant,

$$\mathcal{R}(D_1, D_2) = \tilde{\mathcal{R}}(D_1, D_2).$$

Proof of Theorem 3: We prove inclusions in two directions.

The inclusion $\tilde{\mathcal{R}}(D_1, D_2) \subseteq \mathcal{R}(D_1, D_2)$: If the pair $(\tilde{R}_1 + \tilde{R}_2, \tilde{R}_1) \in \tilde{\mathcal{R}}(D_1, D_2)$, then there exists a code operating at rates $(\tilde{R}_1 + \epsilon, \tilde{R}_2 + \epsilon)$ in the SI-scalable problem achieving the distortion pair $(D_1 + \epsilon, D_2 + \epsilon)$, for any $\epsilon > 0$. Clearly this code can be used for the cascade communication problem, resulting in a code of rates $R_1 = \tilde{R}_1 + \tilde{R}_2 + 2\epsilon$, and $R_2 = \tilde{R}_1 + \epsilon$, with the same distortions. Since the rate-distortion regions for both systems are closed sets, it follows that $(\tilde{R}_1 + \tilde{R}_2, \tilde{R}_1) \in \mathcal{R}(D_1, D_2)$.

The inclusion $\tilde{\mathcal{R}}(D_1, D_2) \supseteq \mathcal{R}(D_1, D_2)$: Suppose $(R_1, R_2) \in \mathcal{R}(D_1, D_2)$, which implies the existence of an n(depending onϵ)-length block code of rate $(R_1 + \epsilon, R_2 + \epsilon)$ in the cascade communication system with resulting distortions $(D_1 + \epsilon, D_2 + \epsilon)$, for any $\epsilon > 0$; in what follows, we will suppress the dependence of n on ϵ for simplicity. Let the messages (for this n-length block code) sent on the first link and the second link be S and T, respectively. Since Z is a constant, T is a function of S and so the following inequalities clearly hold,

$$n(R_2 + \epsilon) \geq H(T) \tag{19}$$

$$n(R_1 + \epsilon) \geq H(S) = H(S, T) = H(T) + H(S|T). \tag{20}$$

We construct a code for the SI-scalable problem from this code. Consider the encoding of m blocks, each of length n. The message $T^m \triangleq (T_1, T_2, ..., T_m)$ can be encoded losslessly on the common link with arbitrarily small error probability P_e^T with a rate $\frac{1}{n}H(T)$, when m is sufficiently large. Since the source is i.i.d., the messages (S_i, T_i) are also i.i.d. Conditioned on T^m, a rate of $\frac{1}{n}H(S|T)$ assures S^m can be transmitted almost losslessly[2], with arbitrarily small error probability P_e^S. Thus this super-code construction is able to achieve the distortion pairs arbitrarily close to $(D_1 + \epsilon, D_2 + \epsilon)$, with rates arbitrarily close to $(\frac{1}{n}H(T), \frac{1}{n}H(S|T))$. This implies that $(\frac{1}{n}H(S), \frac{1}{n}H(T)) \in \tilde{\mathcal{R}}(D_1, D_2)$. The fact that the rate-distortion region is a closed set and (19),(20) imply $(R_1, R_2) \in \tilde{\mathcal{R}}(D_1, D_2)$ and this completes the proof. □

When Z is not a constant, this equivalence between the rate-distortion regions of the two problems does not hold. We demonstrate this by a counterexample. Let (X, Y) be a doubly symmetric binary stream with Y obtained by passing X though a BSC(p). Let $Z = Y$, and we thus have that $X \leftrightarrow Y \leftrightarrow Z$ is indeed a Markov string. Consider the case where $D_1 = 0$ and $D_2 > 0$. The optimal scheme in the cascade problem corresponds to the encoder encoding X losslessly to the relay with rate $R_1 = H(X|Z)$, and the relay using a conditional codebook to encode X with rate $R_2 = R_{X|Y}(D_2)$, where as before $R_{X|Y}(D_2)$ is the conditional rate-distortion function with common side-information Y. In the side-information scalable

[2]This "almost lossless" can be either in the traditional Shannon sense, or in the sense that the Hamming distortion goes to zero.

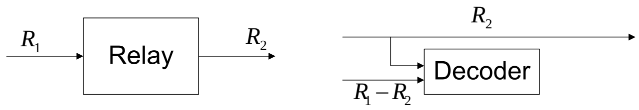

Fig. 4. Equivalence between two systems.

problem, one *cannot* use a conditional codebook at the encoder for decoder 2, which corresponds to the end terminal; in fact the rate on the common link, $\tilde{R}_1$ is lower bounded by $R_{WZ}(D_2)$ with decoder side-information Y. Since it is known that for the DSBS, Wyner-Ziv coding causes rate loss compared with conditional coding, i.e., $R_{WZ}(D_2) > R_{X|Y}(D_2)$ (except for the zero rate case) [9], the equivalence of the rate regions does not hold for this case.

When the relays do not have any side-information, the equivalence between the cascade communication system and the successive refinement system holds in quite general settings. In fact the systems in question can be lossy or lossless, and the source at the encoder can be single source or multi-source. This equivalence holds as long as the rate (or rate distortion) regions of the two systems are closed sets. The link incoming at the relay in the cascade system can be decomposed into two links, a "public" link connecting to the relay and to the further relays/decoders, and a "private link" connecting only to this relay (see Fig. 4). The more complicated relay model can thus be simplified to a "thin relay", which suggests a possible reduction of the computational burden at the relay.

When no side-information exists, the system in consideration reduces to the system considered in [1]. By the above equivalence, it is seen that the rate-distortion region is identical to that of the well-known successive refinement problem [10], [4]. Both problems are solved and their rate-distortion regions are indeed characterized by the same expression. It is now clear that this is not a coincidence, but because of the aforementioned equivalence.

VI. The Gaussian Quadratic Case: Both Switches Open

We consider the case where (X, Y, Z) are jointly Gaussian with $X \sim \mathcal{N}(0, \sigma_X^2)$ and (Y, Z) physically degraded with respect to X. In particular, $Y = X + N_2$ and $Z = X + N_2 + N_2' = X + N_1$, where $N_2 \sim \mathcal{N}(0, \sigma_{N_2}^2)$ and $N_2' \sim \mathcal{N}(0, \sigma_{N_2'}^2)$ are independent. Let $\sigma_{N_1}^2 = \sigma_{N_2}^2 + \sigma_{N_2'}^2$. Only the regime where $D_1 < \sigma_{X|Z}^2$ and $D_2 < \sigma_{X|Y}^2$ is examined here, because otherwise the rate-region is trivial.

Define the threshold distortion D_2^* as

$$D_2^* = \frac{D_1 \sigma_{N_1}^2 \sigma_{N_2}^2}{D_1(\sigma_{N_1}^2 - \sigma_{N_2}^2) + \sigma_{N_1}^2 \sigma_{N_2}^2}$$

Intuitively, D_2^* is a threshold dividing two distortion regimes, corresponding respectively, to the relay and the end terminal having the more stringent distortion requirement. The analysis is done separately for the two regimes of $D_2 \geq D_2^*$ and $D_2 < D_2^*$.

A. The regime $D_2 \geq D_2^*$

In this regime, the random variable U is chosen to satisfy the end decoder, and W is a refinement random variable to satisfy the constraint at the relay. V is chosen to be a constant. Decompose X as $X = W + S_1$ where $W \sim \mathcal{N}(0, \sigma_W^2)$ and $S_1 \sim \mathcal{N}(0, \sigma_{S_1}^2)$ are independent. Further decompose W as $W = U + S_2$ where $U \sim \mathcal{N}(0, \sigma_U^2)$ and $S_2 \sim \mathcal{N}(0, \sigma_{S_2}^2)$ are independent; S_2 is independent of S_1. Let $\sigma_{S_1}^2 = \frac{D_1 \sigma_{N_1}^2}{\sigma_{N_1}^2 - D_1}$ and $\sigma_{S_2}^2 = \frac{D_2 \sigma_{N_2}^2}{\sigma_{N_2}^2 - D_2} - \sigma_{S_1}^2$. It follows $\mathrm{Var}(X|W, Z) = D_1$ and $\mathrm{Var}(X|U, Y) = D_2$. The relay forms an MMSE estimate of X from (W, Z), while the end terminal forms an MMSE estimate of X from (U, Y). Using the fact that $U \leftrightarrow W \leftrightarrow X \leftrightarrow Y \leftrightarrow Z$, the following rates are achievable:

$$\begin{aligned} R_1 &= I(X; U, W|Z) = I(X; W|Z) \\ &= \frac{1}{2} \log \frac{\sigma_{X|Z}^2}{D_1} = R_{WZ}(D_1) \leq R_{HB}(D_1, D_2) \\ R_2 &= I(X; U|Y) = \frac{1}{2} \log \frac{\sigma_{X|Y}^2}{D_2} = R_{X|Y}(D_2). \end{aligned}$$

It is seen that the outer bound for the rate pair (R_1, R_2) is achieved and therefore the scheme is optimal.

B. The regime $D_2 < D_2^*$

In this regime, the description U satisfies the relay and V is a refinement random variable to satisfy the terminal decoder. W is chosen to be a constant. Choose $U = X + S_1$ and $V = X + S_2$ where S_1, S_2 are independent zero-mean Gaussian random variables with variance of $\sigma_{S_1}^2$ and $\sigma_{S_2}^2$, respectively. such that $\mathrm{Var}(X|U, Z) = D_1$ and $\mathrm{Var}(X|U, V, Y) = D_2$. This implies that their variances $\sigma_{S_1}^2$ and $\sigma_{S_2}^2$ satisfy

$$\frac{1}{\sigma_X^2} + \frac{1}{\sigma_{S_1}^2} + \frac{1}{\sigma_{N_1}^2} = \frac{1}{D_1}, \qquad \frac{1}{\sigma_X^2} + \frac{1}{\sigma_{S_1}^2} + \frac{1}{\sigma_{S_2}^2} + \frac{1}{\sigma_{N_2}^2} = \frac{1}{D_2}$$

In the distortion regime we are interested in, a solution to $\sigma_{S_1}^2$ and $\sigma_{S_1}^2$ is guaranteed to exist. The relay forms an MMSE estimate of X from (U, Z) and the end terminal forms an MMSE estimate of X from (U, V, Y). With this choice, the following rate is achievable,

$$\begin{aligned}
R_1 &= I(X; U|Z) + I(X; V|U, Y) \\
&= h(X|Z) - h(X|U, Z) + h(X|U, Y) - h(X|U, V, Y) \\
&= \frac{1}{2}\log(\frac{\sigma^2_{X|Z}}{D_1}) + \frac{1}{2}\log(\frac{\sigma^2_{X|U,Y}}{D_2}) \\
&= \frac{1}{2}\log(\frac{\sigma^2_{X|Z}}{D_1}) + \frac{1}{2}\log(\frac{\sigma^2_{X|Y}\sigma^2_{S_1}}{D_2(\sigma^2_{X|Y} + \sigma^2_{S_1})}) \\
&= R_{HB}(D_1, D_2),
\end{aligned}$$

where the results in [3] is used for the last step. For rate R_2, we have

$$\begin{aligned}
R_2 &= I(X; U|Y) + I(X; V|U, Y) \\
&= h(X|Y) - h(X|U, Y) + h(X|U, Y) - h(X|U, V, Y) \\
&= \frac{1}{2}\log(\frac{\sigma^2_{X|Y}}{\sigma^2_{X|U,Y}}) + \frac{1}{2}\log(\frac{\sigma^2_{X|U,Y}}{D_2}) \\
&= \frac{1}{2}\log(\frac{\sigma^2_{X|Y}}{D_2}) = R_{X|Y}(D_2).
\end{aligned}$$

The outer bound for the rate pair (R_1, R_2) is achieved and therefore the scheme is optimal.

VII. The Gaussian Quadratic Case: Both Switches Closed

As in the previous section, only the regime where $D_1 < \sigma^2_{X|Z}$ and $D_2 < \sigma^2_{X|Y}$ is examined. Define the threshold distortion D_2^* as

$$D_2^* = \frac{D_1\sigma^2_{N_1}\sigma^2_{N_2}}{D_1(\sigma^2_{N_1} - \sigma^2_{N_2}) + \sigma^2_{N_1}\sigma^2_{N_2}}$$

and the threshold distortion D_2^{**} as

$$D_2^{**} = D_1 - \frac{\sigma^4_X(\sigma^2_{N_1} - \sigma^2_{N_2})}{(\sigma^2_X + \sigma^2_{N_2})(\sigma^2_X + \sigma^2_{N_1})}$$

A. *The regime* $D_2 \geq D_2^*$

The achievability scheme is similar to the one described in the previous section for the same distortion regime. The following rates are achievable

$$R_1 = \frac{1}{2}\log\frac{\sigma^2_{X|Z}}{D_1} = R_{X|Z}(D_1)$$
$$R_2 = \frac{1}{2}\log\frac{\sigma^2_{X|Y}}{D_2} = R_{X|Y}(D_2)$$

It is seen that the outer bound for the rate pair (R_1, R_2) is achieved and therefore the scheme is optimal.

B. *The regime* $D_2 \leq D_2^{**}$

We use the scheme described in [5] for their problem setup. In this regime, the optimal description for the end decoder also satisfies the relay. The description U (lossily) encodes the "innovation term" $X - \mathbb{E}[X|Y]$ to within a distortion of D_2 and is therefore independent of (Y, Z). The variables V, W are chosen to be constants. The variable U is described by $U = \alpha X + \beta Y + N_U$ where α, β are constants and $N_U \sim \mathcal{N}(0, 1)$. The constant α is chosen so that $\text{Var}(X|U, Y) = D_1$. Evaluating $\text{Var}(X|U, Y)$,

$$\begin{aligned}
\text{Var}(X|U, Y) &= \text{Var}(X|\alpha X + \beta Y + N_U, X + N_2) \\
&= \text{Var}(X|\alpha X + N_U, X + N_2)
\end{aligned}$$

Therefore α is chosen to satisfy

$$\frac{1}{\sigma^2_X} + \alpha^2 + \frac{1}{\sigma^2_{N_2}} = \frac{1}{D_2}$$

Notice that the value of β does not affect $\text{Var}(X|U, Y)$. Furthermore U is independent of Y iff $\mathbb{E}[UY] = \mathbb{E}[U]\mathbb{E}[Y] = 0$.

$$\begin{aligned}
\mathbb{E}(UY) &= \mathbb{E}[(\alpha X + \beta Y + N_U)Y] \\
&= \alpha + \beta(1 + \sigma^2_{N_2})
\end{aligned}$$

$\mathbb{E}[UY]$ is 0 if $\beta = -\frac{\alpha}{(1+\sigma^2_{N_2})}$. The rate on the first link is given by

$$\begin{aligned}
R_1 &= I(X, Y; U|Z) \\
&= I(X, Y, Z; U) - I(Z; U) \\
&= I(X, Y; U) = h(U) - h(U|X, Y) \\
&= \frac{1}{2}\log\left[(\alpha + \beta)^2\sigma^2_X + \beta^2\sigma^2_{N_2} + 1\right] \\
&= \frac{1}{2}\log\frac{\sigma^2_{X|Y}}{D_2}
\end{aligned}$$

The rate on the second link is given by

$$\begin{aligned}
R_2 &= I(X; U|Y) = I(X, Y; U) - I(Y; U) \\
&= I(X, Y; U) = \frac{1}{2}\log\frac{\sigma^2_{X|Y}}{D_2}
\end{aligned}$$

It is easy to see that the rates on the individual links are optimum if the distortion constraint at the relay is also satisfied. This happpens when

$$\begin{aligned}
D_1 &\geq \text{Var}(X|U, Z) \\
&= D_2 + \frac{\sigma^4_X(\sigma^2_{N_1} - \sigma^2_{N_2})}{(\sigma^2_X + \sigma^2_{N_1})(\sigma^2_X + \sigma^2_{N_2})}
\end{aligned}$$

C. The regime $D_2^{**} < D_2 < D_2^{*}$

In [5], the authors analyze the rate distortion tradeoff for the Gaussian Kaspi problem setup with physically degraded side informations. The lower bound to the rate in their problem setup is a lower bound to the rate on the first link for our problem setup, which we obtain in manner similar for the case with no encoder side information.

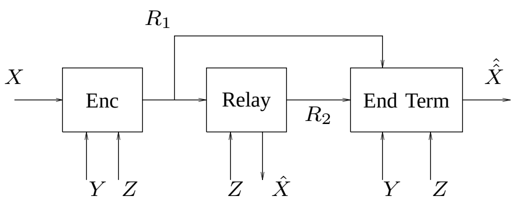

Fig. 5. Lower bound for R_1

We let the terminal decoder have access to the message sent on the first link, as well as the side-information Z. In this new setup (see Fig. 5), the rate R_2 can be set to zero without changing the rate R_1 for the same reason as before. Removing the second link leads to the "conditional (on Z) version" of the problem setup considered by Kaspi, with the decoders having access to the degraded side-informations Z and (Y, Z), respectively, and the encoder having access to the sideinformation (Y, Z). The computation of the rate distortion tradeoff has been done in [5]. Finding the rate distortion tradeoff is equivalent to finding the rate distortion tradeoff for the original (unconditioned version) of the Kaspi setup with the source and side-information stream $(\tilde{X}, \tilde{Y})$ which are distributed as

$$\tilde{X} \sim \mathcal{N}\left(0, \frac{\sigma_X^2 \sigma_{N_1}^2}{\sigma_X^2 + \sigma_{N_1}^2}\right)$$

and $\tilde{Y} = \tilde{X} + \tilde{N}$ where

$$\tilde{N} \sim \mathcal{N}\left(0, \frac{\sigma_{N_1}^2 \sigma_{N_2}^2}{\sigma_{N_1}^2 - \sigma_{N_2}^2}\right)$$

If we let $\hat{R}(D_1, D_2)$ to be the rate distortion tradeoff for this setup, then the following is true

$$R_1 \geq \hat{R}(D_1, D_2)$$

In the regime $D_2^{**} \leq D_2 \leq D_2^{**}$, $\hat{R}(D_1, D_2)$ is given by

$$\begin{aligned} 2^{2\hat{R}(D_1,D_2)} = &\left[D_2 \sigma_{N_1}^2 \sigma_{N_2}^2 \sigma_X^2 (\sigma_{N_1}^2 \sigma_X^2 - D_1(\sigma_{N_1}^2 + \sigma_X 2^))^2\right] \\ &\Bigg[D_1(\sigma_{N_1}^2 + \sigma_X^2) \\ &(D_2 \sigma_{N_1}^2 \sigma_X^2 (-(D_2 \sigma_{N_1}^2 \sigma_{N_2}^2) + \\ &(-(D_2 \sigma_{N_2}^2) + \sigma_{N_2}^4 + D_2(\sigma_{N_1}^2 - \sigma_{N_2}^2) + \\ &\sigma_{N_1}^2 (\sigma_{N_1}^2 - \sigma_{N_2}^2)) \sigma_X^2) + \\ &D_1 ({D_2}^2 \sigma_{N_1}^2 (\sigma_{N_2}^2 + \sigma_X^2)(\sigma_{N_1}^2 + \sigma_X^2) - \\ &D_2 \sigma_{N_1}^2 \sigma_X^2 (\sigma_{N_2}^4 + 2(\sigma_{N_1}^2 - \sigma_{N_2}^2)\sigma_X^2 + \\ &\sigma_{N_2}^2 (\sigma_{N_1}^2 - \sigma_{N_2}^2 + \sigma_X^2)) + \\ &2(\sigma_{N_1}^2 - \sigma_{N_2}^2)\sigma_X^2 (\sigma_{N_1}^2 + \sigma_X^2) \\ &\sqrt{\frac{(D_1 - D_2) D_2 \sigma_{N_1}^4 (D_1 \sigma_{N_2}^2 + (D_1 - \sigma_{N_2}^2)\sigma_X^2)}{{D_1}^2 (\sigma_{N_1}^2 - \sigma_{N_2}^2)(\sigma_{N_1}^2 + \sigma_X^2)}} \\ &\sqrt{\frac{D_2(\sigma_{N_1}^2 \sigma_X^2 - D_2(\sigma_{N_1}^2 + \sigma_X^2))}{\sigma_{N_1} 2 + \sigma_X^2}}))\Bigg]^{-1}. \end{aligned}$$

This lower bound on R_1 is achievable. The description U satisfies both the relay and the end terminal and is described by $U = \alpha X + \beta Y + N_U$ where the α, β are chosen so $\text{Var}(X|U, Z) = D_1$ and $\text{Var}(X|U, Y) = D_2$. The rate on the first link, given by $R_1 = I(X, Y; U|Z)$ is shown in [5] after a series of involved computations to be *equal* to $\hat{R}(D_1, D_2)$. The rate on the second link is given by

$$\begin{aligned} R_2 &= I(X; U|Y) = h(X|Y) - h(X|U, Y) \\ &= \frac{1}{2} \log \frac{\sigma_{X|Y}^2}{D_2} = R_{X|Y}(D_2) \end{aligned}$$

The scheme achieves optimal rate on the second link.

VIII. The Gaussian Quadratic Case: Switch 1 Closed, Switch 2 Open

The optimum descriptions (U, W) for all the regimes in the previous section satisfied $(U, W) \leftrightarrow (X, Y) \leftrightarrow Z$. Therefore, the rate distortion region when the encoder does not have access to the sideinformation Z is the same as the rate distortion region presented in the previous section for the case when the encoder has access to both the side informations.

IX. The Gaussian Quadratic Case: Switch 1 Open, Switch 2 Closed

In the previous sections, for the regime $D_2 \geq D_2^*$, the optimal descriptions (U, W) satisfy $(U, W) \leftrightarrow X \leftrightarrow (Y, Z)$. The tradeoff in this regime therefore is the same as in the previous cases. For the regime $D_2 < D_2^*$, we lower bound the rate on the first link by making the side information Z and the message on the first link available at the terminal decoder. For the same reasons as before, the rate R_2 can be set to zero without changing the rate R_1. Removing the second link leads to the "conditional(on Z) version" of the problem setup considered by Heegard-Berger. Finding the rate distortion tradeoff for this problem is equivalent to finding the rate distortion tradeoff for the original Heegard-Berger setup with the (conditioned on Z) source and side-information stream $(\tilde{X}, \tilde{Y})$ described in the previous section. Let us call this tradeoff $\hat{R}(D_1, D_2)$. Then,

$$R_1 \geq \hat{R}(D_1, D_2)$$

Evaluating $\hat{R}(D_1, D_2)$ yields the rate distortion tradeoff for the Heegard-Berger problem with *no side-information* at the encoder and the side-informations Y and Z respectively at the two decoders. This shows that the region does not expand when only the degraded side-information Z is made available to the encoder as compared to the case with no encoder side information.

X. The Doubly Symmetric Binary Source

We consider the case when X is a binary symmetric source sequence and Y is the side-information obtained by passing X through a BSC(p) with $p < 0.5$. Z does not exist. Unlike in the Gaussian case, we are only able to show the optimality of the achievable region in certain regimes.

Only the case when $D_1 < \frac{1}{2}$ and $D_2 < p$ is considered here. Recall that the rate-distortion function of the Wyner-Ziv problem corresponding to X, Y being the source and the side-information, respectively, is the lower convex hull of the curve $\{d \in [0, p] : (h(p * d) - h(d), d)\}$ and the point $(0, p)$. Therefore there is a certain distortion D_c larger than which time-sharing with the zero rate point should be employed [9]. The inner and outer bounds can be shown to coincide in the regime $D_1 \leq D_2 \leq D_c$ and the regime $D_1 \leq D_c < D_2$.

A. *The regime* $D_1 \leq D_2 \leq D_c$

Let U be a random variable which achieves the Wyner-Ziv bound for the end decoder, and W be the refinement variable for the relay. In particular, choose $W = X + Z_1$ and $U = X + Z_1 + Z_2$ where $Z_1 \sim$ Bern(D_1) and $Z_2 \sim$ Bern(q) are independent and q is such that $q * D_1 = D_2$, where $*$ denotes the convolution operation, i.e, $x * y = x(1-y) + (1-x)y$. The following rates are achievable,

$$R_1 = 1 - h(D_1), \quad R_2 = h(p * D_2) - h(D_2).$$

It is seen that the individual rates are optimal.

B. *The regime* $D_1 \leq D_c < D_2$

Let λ be such that $D_2 = \bar{\lambda} D_c + \lambda p$. Let U be a random variable which achieves the Wyner-Ziv bound for the end decoder and W be the refinement random variable for the relay. In particular, let $W = X \oplus Z_1$ and define $U' = X \oplus Z_1 \oplus Z_2$, where $Z_1 \sim$ Bern(D_1) and $Z_2 \sim$ Bern(q) are independent and q is such that $q * D_1 = D_c$. Let U be the output of a BEC(λ) with input U'. The following rates are achievable,

$$R_1 = 1 - h(D_1), \quad R_2 = \bar{\lambda}(h(p * D_c) - h(D_c)).$$

It is seen that the individual rates are optimal.

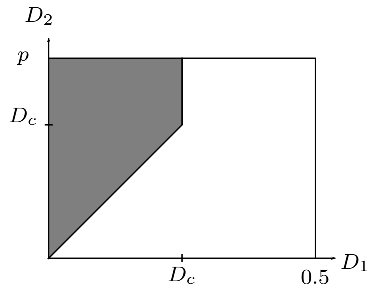

Fig. 6. The shaded region is the regime where the inner and outer bounds coincide.

XI. Conclusion and Future Work

We investigated source coding for a cascade communication system with side-informations, where the source and the side-informations are physically degraded. The provided inner bound and outer bound do not coincide in general, however, they do provide a complete solution for the jointly Gaussian source under MSE distortion measure. A connection is made between the CC-SI problem and SI-scalable problem, which can be used to show the equivalence of the rate-distortion regions between the successive refinement system and Yamamoto's cascading communication system. Possible extension of the current work includes the case when the side-informations are stochastically degraded, and when the side informations do not have any degradation structure.

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