

On successive refinement of diversity for fading ISI channels

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Abstract

Rate and diversity impose a fundamental trade-off in communications. This trade-off was investigated for flat-fading channels in [16] as well as for Inter-symbol Interference (ISI) channels in [1]. A different point of view was explored in [13] where high-rate codes were designed so that they have a high-diversity code embedded within them. These diversity embedded codes were investigated for flat fading channels both from an information-theoretic viewpoint [5] and from a coding theory viewpoint in [2]. In this paper, we explore the use of diversity embedded codes for inter-symbol interference channels. In particular the main result of this paper is that the diversity multiplexing trade-off for fading MISO/SIMO/SISO ISI channels is indeed successively refinable. This implies that for fading ISI channels with a single degree of freedom one can embed a high diversity code within a high rate code without any performance loss (asymptotically). This is related to a deterministic structural observation about the asymptotic behavior of frequency response of channel with respect to fading strength of time domain taps as well as a coding scheme to take advantage of this observation.

I. INTRODUCTION

The classical approach towards code design for channels is to maximize the data rate given a desired level of reliability. The classical outage formulation divides the set of channel realizations into an outage set \mathcal{O} and a non-outage set $\overline{\mathcal{O}}$: it requires that a code has to be designed such that the transmitted message can be decoded with arbitrary small error probability on all the channels in the non-outage set. Since the

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code must work for all such channels, the data rate is limited by the worst channel in the non-outage set. Note that in this scenario, the communication strategy cannot take advantage of the opportunity when the channel happens to be stronger than the worst channel in the non-outage set. For this classical approach, a seminal result in [16] showed that there exists a fundamental trade-off between diversity (error probability) and multiplexing (rate). This was characterized in the high SNR regime for flat fading channels with multiple transmit and multiple receive antennas (MIMO) [16]. This D-M trade-off has been extended to several cases including scalar (SISO) fading ISI channels [1]. The presence of ISI gives significant improvement of the diversity order. In fact, for the SISO case the improvement was equivalent to having multiple receive antennas equal to the number of ISI taps [1].

Diversity embedded coding takes advantage of the good channel realizations by an opportunistic coding strategy [3]. Although the focus is on two levels of diversity, the results can be easily generalized to arbitrary number of levels. Consider two information streams with \mathcal{H} denoting the message set from the first information stream and \mathcal{L} denoting that from the second information stream. Diversity embedded codes encode the streams such that the high-priority stream (\mathcal{H}) is decoded with arbitrary small error probability whenever the channel is not in outage (\mathcal{O}) and in addition the lower-priority stream is decoded whenever the channel is in a set $\mathcal{G} \subset \overline{\mathcal{O}_H}$ of good channels (see Figure 1).

In this paper we explore the performance of diversity embedded codes [5] over ISI channels with single degree of freedom *i.e.*, $\min(M_t, M_r) = 1$. The rates for the higher and lower priority message sets, as a function of SNR , are respectively $R_H(SNR)$ and $R_L(SNR)$. Consider transmission over a channel for which $(r, D^{opt}(r))$ is the optimal single-layer diversity-multiplexing point corresponding to the channel. After transmission the decoder jointly decodes the two message sets and we can define two error probabilities, $P_e^H(SNR)$ and $P_e^L(SNR)$, which denote the average error probabilities for message sets \mathcal{H} and \mathcal{L} respectively. We want to characterize the tuple (r_H, D_H, r_L, D_L) of rates and diversities channel that are achievable where,

$$D_H = \lim_{SNR \rightarrow \infty} -\frac{\log P_e^H(SNR)}{\log(SNR)}, \quad r_H = \lim_{SNR \rightarrow \infty} \frac{R_H(SNR)}{\log(SNR)}$$

$$D_L = \lim_{SNR \rightarrow \infty} -\frac{\log P_e^L(SNR)}{\log(SNR)}, \quad r_L = \lim_{SNR \rightarrow \infty} \frac{R_L(SNR)}{\log(SNR)}.$$

If viewed as a single-layer code, the diversity embedded code achieves rate-diversity pairs (r_H, D_H) and

$(r_H + r_L, D_L)$, where it is assumed that $D_H \geq D_L$. Since it is not possible to beat the single-layer D-M trade-off, note that necessarily $D_H \leq D^{opt}(r_H)$ and $D_L \leq D^{opt}(r_H + r_L)$.

In [5] it was shown that when we have one degree of freedom (one transmit many receive or one receive many transmit antennas) the D-M trade-off, for a flat-fading channel, was successively refinable. That is, the high priority scheme (with higher diversity order) can attain the optimal diversity-multiplexing (D-M) performance as if the low priority stream was absent. This property of successive refinement is illustrated in Figure 2. However, the low priority scheme (with lower diversity order) attains the same D-M performance as that of the aggregate rate of the two streams. When there is more than one degree of freedom (for example, parallel fading channels) such a successive refinement property does not hold [4].

Since the Fourier basis is the eigenbasis for linear time invariant channels we can decompose the transmission into a set of parallel channels. Since it is known that the D-M trade-off for parallel fading channels is not successively refinable [4], it is tempting to expect the same for fading ISI channels. The main result in this paper demonstrates that for fading ISI channels with one degree of freedom (SISO/SIMO/MISO) the D-M trade-off is indeed successively refinable. At first this result might seem surprising, but the correlations of the fading across the parallel channels cause the difference in the behavior. In particular, in Section III we demonstrate the following deterministic observation. For ISI channels with a finite impulse response, we show that there is enough correlation in the frequency domain such that most frequency domain taps behave like the maximum of the time-domain taps, in the high SNR regime. This property causes the difference in independent parallel channels and the ISI channels.

For the SISO ISI case we showed [9] that uncoded transmission is sufficient to demonstrate successive refinability. For the MISO case we need to develop a coding strategy related to universal codes [10] to obtain our main result. Surprisingly like the flat fading case, the ISI fading channel with single degree of freedom (SISO/SIMO/MISO) is successively refinable. The main result of this paper is stated below.

Theorem 1.1: The diversity multiplexing trade-off for a ν tap point to point MISO/SIMO/SISO ISI channel is successively refinable, *i.e.*, for any multiplexing gains r_H and r_L such that $r_H + r_L \leq \frac{T_s - \nu}{T_s}$ the

achievable diversity orders given by $D_H(r_H)$ and $D_L(r_L)$ are bounded as,

$$(\nu + 1)M_t \left(1 - \frac{T_s}{(T_s - \nu)} r_H \right) \leq D_H(r_H) \leq (\nu + 1)M_t (1 - r_H), \quad (1)$$

$$(\nu + 1)M_t \left(1 - \frac{T_s}{(T_s - \nu)} (r_H + r_L) \right) \leq D_L(r_L) \leq (\nu + 1)M_t (1 - (r_H + r_L)) \quad (2)$$

where T_s is finite and does not grow with SNR.

Note that Theorem 1.1 holds for arbitrary number of levels of diversity, *i.e.*, the diversity multiplexing trade-off is *infinitely divisible*. An implication of Theorem 1.1 is that for MISO/SIMO/SISO fading ISI channels, one can design "ideal" opportunistic codes which adjust to the rate supported by the fading channel without apriori knowing about the the channel. This property could be used for allowing new networking functionalities through opportunistic scheduling [17] as well as wireless multimedia delivery. In summary, we believe that this property and the code construction used to achieve this result could be important for future broadband wireless system design.

The paper is organized as follows. In Section II we formulate the problem statement and present the notation. A crucial structural observation on the behavior of ISI fading channels is established in III. Using these observation, in Section IV, we propose a transmission technique to code across space-time-frequency suitable for fading MISO ISI channels; for such codes with a non-vanishing determinant criterion, we establish the result for MISO ISI channel. We conclude with a short discussion in Section V. Some of the more detailed proofs are provided in the Appendices.

II. PROBLEM STATEMENT

Our focus is on the quasi-static fading ISI channel where we transmit information coded over M_t transmit antennas with M_r antennas at the receiver. Throughout this paper, we assume that the transmitter has no channel state information (CSI), whereas the receiver is able to perfectly track the channel (a common assumption, see for example [15], [11]).

The coding scheme is limited to one quasi-static transmission block of large enough block size $T \geq T_{thr}$ to be specified later. The received vector at time n after demodulation and sampling can be written as

$$\mathbf{y}[n] = \mathbf{H}_0 \mathbf{x}[n] + \mathbf{H}_1 \mathbf{x}[n - 1] + \dots + \mathbf{H}_\nu \mathbf{x}[n - \nu] + \mathbf{z}[n] \quad (3)$$

where $\mathbf{y} \in \mathbb{C}^{M_r \times 1}$ is the received vector at time n , $\mathbf{H}_l \in \mathbb{C}^{M_r \times M_t}$ represents the l^{th} matrix tap of the MIMO ISI channel, $\mathbf{x}[n] \in \mathbb{C}^{M_t \times 1}$ is the space-time coded transmission vector at time n with transmit power constraint P and $\mathbf{z} \in \mathbb{C}^{M_r \times 1}$ is assumed to be additive white (temporally and spatially) Gaussian noise with variance σ^2 . We use SNR to represent the signal to noise ratio for the period of communication. The matrix \mathbf{H}_l consists of fading coefficients h_{ij} which are i.i.d. $\mathcal{CN}(0, 1)$ and fixed for the duration of the block length (T). Let $h_i^{(p,q)}$ represent the i^{th} tap coefficient between the p^{th} receive antenna and the q^{th} transmit antenna, $x^{(q)}[k]$ and $y^{(p)}[n]$ be the symbol transmitted on the q^{th} transmit antenna and the symbol received at the p^{th} receive antenna in the n^{th} time instant, respectively.

Also, let $\mathbf{x}_{[a,b]}^{(q)}$ and $y_{[a,b]}^{(p)}$ represent the symbols transmitted on the q^{th} transmit antenna and received at the p^{th} receive antenna over the time period a to b , *i.e.*,

$$\mathbf{y}_{[0, T_s-1]}^{(p)} = \begin{bmatrix} y^{(p)}[0] & y^{(p)}[1] & \dots & y^{(p)}[T_s - 1] \end{bmatrix}^t.$$

Consider a sequence of coding schemes with transmission rate as a function of SNR given by $R(SNR)$ and an average error probability of decoding $P_e(SNR)$. Analogous to [16] we define the multiplexing rate r and the diversity order D as follows,

$$D = \lim_{SNR \rightarrow \infty} -\frac{\log P_e(SNR)}{\log(SNR)}, \quad r = \lim_{SNR \rightarrow \infty} \frac{R(SNR)}{\log(SNR)}. \quad (4)$$

We use the special symbol \doteq to denote exponential equality *i.e.*, we write $f(SNR) \doteq SNR^b$ to denote

$$\lim_{SNR \rightarrow \infty} \frac{\log f(SNR)}{\log(SNR)} = b$$

and $\dot{\leq}$ and $\dot{\geq}$ are defined similarly. We use the following definition for successive refinability.

Definition 2.1: [6] A channel is said to be successively refinable if the diversity-multiplexing trade-off curve for transmission is successively refinable, *i.e.*, for any multiplexing gains r_H and r_L such that $r_H + r_L \leq \min(M_t, M_r)$, the diversity orders

$$D_H = D^{opt}(r_H), \quad D_L = D^{opt}(r_H + r_L) \quad (5)$$

are achievable, where $D^{opt}(r)$ is the optimal diversity order of the channel.

The concept of successive refinability can be visualized as in Figure 2. For codes that are successively refinable this definition implies that one can perfectly embed a high diversity code within a high rate code

i.e., the high-priority can attain the optimal diversity performance as though the low-priority stream were not there and yet the diversity performance of the low priority stream is the same as the optimal diversity of a stream with the aggregate rate of the two streams.

From an information-theoretic point of view [5] focused on the case when there is one degree of freedom, (*i.e.*, $\min(M_t, M_r) = 1$), and transmission over a flat fading Rayleigh channel. In that case if we consider $D_H \geq D_L$ without loss of generality, it was established [5] that the channel is successively refinable. This implies that for channels with a single degree of freedom $\min(M_t, M_r) = 1$, we can design ideal *opportunistic* codes and that the D-M trade-off for SIMO/MISO are successively refinable. The question of successive refinability was further investigated in [4] for K parallel i.i.d channels, the simplest of the channels with multiple degrees of freedom, and it was shown that the channel is not successively refinable. In particular, if we desire the optimal performance for the higher layer stream (r_H) then there is a loss of diversity of $(K - 1)r_H$ due to the embedding and therefore the K parallel i.i.d. channel is not successively refinable.

The diversity multiplexing trade-off for a *scalar* fading ISI channel was established in [1], and the result is summarized below.

Theorem 2.2: [1] The diversity multiplexing trade-off for transmission over a SISO ISI channel with $\nu + 1$ taps for transmission over a period of time T_s assuming perfect channel knowledge only at the receiver for $0 \leq r \leq \frac{T_s - \nu}{T_s}$ is bounded by

$$(\nu + 1) \left(1 - \frac{T_s}{T_s - \nu} r \right) \leq D_{isi}(r) \leq (\nu + 1)(1 - r). \quad (6)$$

The D-M trade-off for the SIMO channel can also be easily obtained using techniques similar to the proof of this result. Since the D-M trade-off for parallel independent channels is not successively refinable and given the derivation of the SISO ISI trade-off it might be tempting to conclude that the D-M trade-off for the ISI channel is not successively refinable. We will show in this paper that the ISI channel is successively refinable by utilizing the fact that correlations exist across these sets of independent parallel channels.

III. STRUCTURAL OBSERVATION

In this section we make a deterministic structural observation relating the value of the taps in frequency domain to the value of the taps in time domain. To make the observation we consider the MIMO model in (3) and consider the specific transmission scheme as in the previous section where we transmit for a period of $T_s - \nu$ time instants and pad it with ν zero symbols. We refer to this zero padded block of length T_s as one symbol. The received symbols over the period of T_s can be written as,

$$\underbrace{\begin{bmatrix} \mathbf{y}[0] \\ \mathbf{y}[1] \\ \vdots \\ \mathbf{y}[T_s - \nu - 1] \\ \dots \\ \mathbf{y}[T_s - 1] \end{bmatrix}}_{\mathbf{Y}} = \underbrace{\begin{bmatrix} \mathbf{H}_0 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{H}_\nu & \dots & \mathbf{H}_2 & \mathbf{H}_1 \\ \mathbf{H}_1 & \mathbf{H}_0 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{H}_\nu & \dots & \mathbf{H}_2 \\ \vdots & \vdots & & & \dots & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{H}_\nu & \mathbf{H}_{\nu-1} & \dots & \mathbf{H}_1 & \mathbf{H}_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \\ \mathbf{x}[T_s - \nu - 1] \\ \mathbf{0}_{\nu \times 1} \end{bmatrix}}_{\mathbf{W}} + \underbrace{\begin{bmatrix} \mathbf{z}[0] & \mathbf{z}[1] & \dots & \mathbf{z}[T_s - \nu - 1] & \dots & \mathbf{z}[T_s - 1] \end{bmatrix}}_{\mathbf{Z}} \quad (7)$$

where $\mathbf{Y} \in \mathbb{C}^{T_s M_r \times 1}$, $\mathbf{H} \in \mathbb{C}^{T_s M_r \times T_s M_t}$, $\mathbf{W} \in \mathbb{C}^{T_s M_t \times 1}$, $\mathbf{Z} \in \mathbb{C}^{T_s M_r \times 1}$. Rearranging and permuting the rows and columns of equation (7), we get

$$\begin{bmatrix} \mathbf{y}_{[0, T_s - 1]}^{(1)} \\ \mathbf{y}_{[0, T_s - 1]}^{(2)} \\ \vdots \\ \mathbf{y}_{[0, T_s - 1]}^{(M_r)} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{H}^{(1,1)} & \mathbf{H}^{(1,2)} & \dots & \mathbf{H}^{(1, M_t)} \\ \mathbf{H}^{(2,1)} & \mathbf{H}^{(2,2)} & \dots & \mathbf{H}^{(2, M_t)} \\ \vdots & \vdots & & \vdots \\ \mathbf{H}^{(M_r, 1)} & \mathbf{H}^{(M_r, 2)} & \dots & \mathbf{H}^{(M_r, M_t)} \end{bmatrix}}_{\mathbf{H}_{circ}} \begin{bmatrix} \mathbf{x}_{[0, T_s - 1]}^{(1)} \\ \mathbf{x}_{[0, T_s - 1]}^{(2)} \\ \vdots \\ \mathbf{x}_{[0, T_s - 1]}^{(M_t)} \end{bmatrix} + \mathbf{Z} \quad (8)$$

where $\mathbf{H}^{(p,q)}$ are circulant matrices given by

$$\mathbf{H}^{(p,q)} = circ\{h_0^{(p,q)}, 0, \dots, 0, h_\nu^{(p,q)}, \dots, h_2^{(p,q)}, h_1^{(p,q)}\}. \quad (9)$$

Since the $\mathbf{H}^{(p,q)}$ are circulant matrices they can be written using the frequency-domain notation as $\mathbf{H}^{(p,q)} = \mathbf{Q}\mathbf{\Lambda}^{(p,q)}\mathbf{Q}^*$ where $\mathbf{\Lambda}^{(p,q)}$ are diagonal matrices with elements given by

$$\mathbf{\Lambda}^{(p,q)} = diag\left\{ \lambda_k^{(p,q)} : \lambda_k^{(p,q)} = \sum_{l=0}^{\nu} h_l^{(p,q)} e^{-\frac{2\pi j}{T_s} kl} \right\} \quad \text{for } k = \{0, \dots, (T_s - 1)\}. \quad (10)$$

To get an intuition of the result consider the polynomial

$$\lambda^{(p,q)}(z) = \sum_{m=0}^{\nu} h_m^{(p,q)} z^m,$$

which evaluates to the k^{th} tap coefficient in the frequency domain for $z = e^{-\frac{2\pi j}{T_s}k}$. Since this is a polynomial of maximum degree ν , if we evaluate the polynomial at $z = e^{-\frac{2\pi j}{T_s}k}$ for $k = \{0, \dots, (T_s - 1)\}$, at most ν values can be zero and at least $T_s - \nu$ values are bounded away from zero. The following lemma formalizes this intuition and relates the asymptotic behaviors of the frequency-domain coefficients to the fading strength of the time-domain taps. This lemma is then used in the remaining sections to show the successive refinement of the ISI trade-off.

Lemma 3.1: Consider the taps in the frequency domain in (10) given by

$$\lambda_k^{(p,q)} = \sum_{l=0}^{\nu} h_l^{(p,q)} e^{-\frac{2\pi j}{T_s}kl}$$

for $k = \{0, \dots, (T_s - 1)\}$, $p \in \{1, \dots, M_r\}$ and $q \in \{1, \dots, M_t\}$. For $\alpha \in (0, 1]$, define the sets $\mathcal{G}^{(p,q)}$, $\mathcal{F}^{(p,q)}(\alpha)$ and $\mathcal{M}(\alpha)$ as

$$\mathcal{G}^{(p,q)} = \{k : |\lambda_k^{(p,q)}|^2 \doteq \max_{l \in \{0, 1, \dots, \nu\}} |h_l^{(p,q)}|^2\}, \quad \mathcal{F}^{(p,q)}(\alpha) = \{k : |\lambda_k^{(p,q)}|^2 \dot{<} SNR^{-\alpha}\}, \quad (11)$$

$$\mathcal{M}(\alpha) = \{\mathbf{h} : |h_i^{(p,q)}|^2 \dot{\leq} SNR^{-\alpha}, \forall i \in \{0, \dots, \nu\}, \forall p \in \{1, \dots, M_r\}, q \in \{1, \dots, M_t\}\}. \quad (12)$$

We have the following relations on the cardinality of these sets:

- (a) Letting $\overline{\mathcal{G}^{(p,q)}}$ represent the complement of the set $\mathcal{G}^{(p,q)}$, we have

$$|\overline{\mathcal{G}^{(p,q)}}| \leq \nu \quad \forall p, q. \quad (13)$$

In other words, at least $T_s - \nu$ of the T_s taps in the frequency domain, for each (p, q) , are (asymptotically) of magnitude $\max(|h_0^{(p,q)}|^2, |h_1^{(p,q)}|^2, \dots, |h_\nu^{(p,q)}|^2)$.

- (b) Given that $\mathbf{H} \in \overline{\mathcal{M}(\alpha)}$, for MISO channel

$$\exists q \in \{1, 2, \dots, M_t\} \text{ s. t. } |\mathcal{F}^{(1,q)}(\alpha)| \leq \nu \quad (14)$$

and for a SIMO channel

$$\exists p \in \{1, 2, \dots, M_r\} \text{ s. t. } |\mathcal{F}^{(p,1)}(\alpha)| \leq \nu \quad (15)$$

Proof: The tap coefficients in the frequency domain are given by,

$$\lambda_k^{(p,q)} = \sum_{m=0}^{\nu} h_m^{(p,q)} e^{-\frac{2\pi j}{T_s} km} \quad k = \{0, \dots, (T_s - 1)\}$$

Defining $\theta = e^{-\frac{2\pi j}{T_s}}$ the above equation can be rewritten as,

$$\begin{aligned} \lambda_k^{(p,q)} &= \sum_{m=0}^{\nu} h_m^{(p,q)} \theta^{km} \quad k = \{0, \dots, (T_s - 1)\} \\ &= \begin{bmatrix} 1 & \theta^k & \dots & \theta^{k\nu} \end{bmatrix} \begin{bmatrix} h_0^{(p,q)} & h_1^{(p,q)} & \dots & h_{\nu}^{(p,q)} \end{bmatrix}^t \end{aligned} \quad (16)$$

Take any set of $(\nu + 1)$ coefficients in the frequency domain and index this set by $\mathcal{K} = \{k_0, \dots, k_{\nu}\}$ and define,

$$\check{\mathbf{\Lambda}}^{(p,q)} = \begin{bmatrix} \lambda_{k_0}^{(p,q)} \\ \lambda_{k_1}^{(p,q)} \\ \vdots \\ \lambda_{k_{\nu}}^{(p,q)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \theta^{k_0} & \dots & \theta^{k_0\nu} \\ 1 & \theta^{k_1} & \dots & \theta^{k_1\nu} \\ \vdots & & \dots & \vdots \\ 1 & \theta^{k_{\nu}} & \dots & \theta^{k_{\nu}\nu} \end{bmatrix}}_{\mathbf{V}} \underbrace{\begin{bmatrix} h_0^{(p,q)} \\ h_1^{(p,q)} \\ \vdots \\ h_{\nu}^{(p,q)} \end{bmatrix}}_{\mathbf{h}} \quad (17)$$

where $\mathbf{V} \in \mathbb{C}^{(\nu+1) \times (\nu+1)}$ is a full rank Vandermonde matrix. Therefore, the inverse of \mathbf{V} exists and we denote it by $\mathbf{V}^{-1} = \mathbf{A}$ and let a_{li} represent the element in the l^{th} row and i^{th} column. From (17) we have,

$$\begin{bmatrix} h_0^{(p,q)} & h_1^{(p,q)} & \dots & h_{\nu}^{(p,q)} \end{bmatrix}^t = \mathbf{V}^{-1} \check{\mathbf{\Lambda}}^{(p,q)}, \quad i.e. \quad h_l^{(p,q)} = \sum_{i=0}^{\nu} a_{li} \lambda_{k_i}^{(p,q)} \quad l = \{0, 1, \dots, \nu\}.$$

Using the Cauchy-Schwartz inequality¹, we get,

$$|h_l^{(p,q)}|^2 = \left| \sum_{i=0}^{\nu} a_{li} \lambda_{k_i}^{(p,q)} \right|^2 \leq \left(\sum_{i=0}^{\nu} |a_{li}|^2 \right) \left(\sum_{i=0}^{\nu} |\lambda_{k_i}^{(p,q)}|^2 \right)$$

Using the fact that T_s is finite and does not grow with SNR it follows that the $\{a_{li}\}$ do not depend on SNR . Therefore, the above inequality can be asymptotically written as

$$|h_l^{(p,q)}|^2 \leq |\lambda_{k_0}^{(p,q)}|^2 + |\lambda_{k_1}^{(p,q)}|^2 + \dots + |\lambda_{k_{\nu}}^{(p,q)}|^2. \quad (18)$$

Note that the above inequality holds for all $h_l^{(p,q)}$, $l = 0, \dots, \nu$. Therefore, we get that for any set of $(\nu + 1)$ coefficients in the frequency domain indexed by $\{k_0, \dots, k_{\nu}\}$,

$$\max_{l \in \{0, 1, \dots, \nu\}} |h_l^{(p,q)}|^2 \leq |\lambda_{k_0}^{(p,q)}|^2 + |\lambda_{k_1}^{(p,q)}|^2 + \dots + |\lambda_{k_{\nu}}^{(p,q)}|^2 \quad (19)$$

¹ $|\mathbf{u}^* \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$.

From the Cauchy-Schwartz inequality note that,

$$\begin{aligned}
|\lambda_{k_0}^{(p,q)}|^2 + |\lambda_{k_1}^{(p,q)}|^2 + \dots + |\lambda_{k_\nu}^{(p,q)}|^2 &= \left| \sum_{m=0}^{\nu} h_m^{(p,q)} \theta^{k_0 m} \right|^2 + \dots + \left| \sum_{m=0}^{\nu} h_m^{(p,q)} \theta^{k_\nu m} \right|^2 \\
&\leq \left(\sum_{m=0}^{\nu} |h_m^{(p,q)}|^2 \right) \left(\sum_{m=0}^{\nu} |\theta^{k_0 m}|^2 + \dots + \sum_{m=0}^{\nu} |\theta^{k_\nu m}|^2 \right) \\
&\doteq (|h_0^{(p,q)}|^2 + |h_1^{(p,q)}|^2 + \dots + |h_\nu^{(p,q)}|^2) \doteq \max_{l \in \{0,1,\dots,\nu\}} |h_l^{(p,q)}|^2
\end{aligned} \tag{20}$$

Combining equations (19) and (20) we get,

$$|\lambda_{k_0}^{(p,q)}|^2 + |\lambda_{k_1}^{(p,q)}|^2 + \dots + |\lambda_{k_\nu}^{(p,q)}|^2 \doteq \max_{l \in \{0,1,\dots,\nu\}} |h_l^{(p,q)}|^2. \tag{21}$$

- We know from (20) that for all k , $|\lambda_k|^2 \leq \max_{l \in \{0,1,\dots,\nu\}} |h_l|^2$. Since, $\mathcal{G}^{(p,q)} = \{i : |\lambda_i^{(p,q)}|^2 \doteq \max_{l \in \{0,1,\dots,\nu\}} |h_l^{(p,q)}|^2\}$,

$$|\lambda_k^{(p,q)}|^2 < \max_{l \in \{0,1,\dots,\nu\}} |h_l^{(p,q)}|^2 \quad \forall k \in \overline{\mathcal{G}^{(p,q)}}.$$

If $|\overline{\mathcal{G}^{(p,q)}}| > \nu$ then there exists a set $\mathcal{K} = \overline{\mathcal{G}^{(p,q)}}$ of size at least $\nu + 1$ such that,

$$|\lambda_k^{(p,q)}|^2 < \max_{l \in \{0,1,\dots,\nu\}} |h_l^{(p,q)}|^2 \quad \forall k \in \mathcal{K}$$

But this is a contradiction to equation (21) and therefore we have $|\overline{\mathcal{G}^{(p,q)}}| \leq \nu$ proving (a).

- For a MISO channel, given that $\mathbf{H} \in \overline{\mathcal{M}(\alpha)}$, we know that there exists at least one (\hat{i}, \hat{q}) pair such that $|h_{\hat{i}}^{(1,\hat{q})}|^2 > \frac{1}{SNR^\alpha}$. For this \hat{q} , since $|\overline{\mathcal{G}^{(1,\hat{q})}}| \leq \nu$ it follows that $|\mathcal{F}^{(1,\hat{q})}(\alpha)| \leq \nu$. A similar argument holds for SIMO as well, proving (b). ■

IV. SUCCESSIVE REFINEMENT OF MISO ISI CHANNEL

We showed in [8] (see [9] for more details on this special case) that superposition of uncoded QAM constellations followed by zero padding is sufficient to derive the DM tradeoff and prove its successive refinability for SISO/SIMO ISI channels. The intuition that was used in deriving the successive refinement of the SISO trade-off for ISI channels was that given that $\mathbf{h} \in \mathcal{M}(1-r)$ at most ν taps in the frequency domain are zero and the remaining are “good” and of the same magnitude. This intuition can also be carried over to show the successive refinability of the SIMO channel with M_r receive antennas and one transmit antenna as well. The proof of successive refinability of the MISO channel requires a more

sophisticated coding strategy as we will show in this section (see also [7]). We will consider the same model as in Section III and consider a slight variant of the scheme where we transmit for a period of $(T_s - \nu)$ followed by padding with ν zeros. The variant is that instead of doing this once we repeat the process of transmission for $(T_s - \nu)$ followed by ν zeros for $T_b = T_s M_t$ symbols for a total communication period of $T = T_b T_s$. The scheme is as shown in the Figure 3.

We rearrange the received T symbols in a matrix form $\mathbf{Y} \in \mathbb{C}^{T_s \times T_b}$ where,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}_{[0, T_s-1]}^{(1)} & \mathbf{y}_{[T_s, 2T_s-1]}^{(1)} & \cdots & \mathbf{y}_{[(T_b-1)T_s, T_b T_s-1]}^{(1)} \end{bmatrix}. \quad (22)$$

Denote $\check{\mathbf{X}}^{(i)} \in \mathbb{C}^{T_s \times T_b}$ to be the symbols transmitted by the i^{th} transmit antenna over the period of communication *i.e.*,

$$\check{\mathbf{X}}^{(i)} = \begin{bmatrix} \mathbf{x}_{[0, T_s-\nu-1]}^{(i)} & \mathbf{x}_{[T_s, 2T_s-\nu-1]}^{(1)} & \cdots & \mathbf{x}_{[(T_b-1)T_s, T_b T_s-\nu-1]}^{(i)} \\ \mathbf{0}_{\nu \times 1} & \mathbf{0}_{\nu \times 1} & \cdots & \mathbf{0}_{\nu \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(i)} \\ \mathbf{0}_{\nu \times T_b} \end{bmatrix},$$

for $i \in \{1, 2, \dots, M_t\}$. Similar to (8) we can rearrange the rows and columns and the received symbols can be written as,

$$\mathbf{Y} = \begin{bmatrix} \mathbf{H}^{(1,1)} & \mathbf{H}^{(1,1)} & \dots & \mathbf{H}^{(1, M_t)} \end{bmatrix} \begin{bmatrix} \check{\mathbf{X}}^{(1)t} & \check{\mathbf{X}}^{(2)t} & \dots & \check{\mathbf{X}}^{(M_t)t} \end{bmatrix}^t + \mathbf{Z} \quad (23)$$

where $\mathbf{H}^{(1,1)}, \dots, \mathbf{H}^{(1, M_t)} \in \mathbb{C}^{T_s \times T_s}$ are circulant matrices given in (9). Decomposing them in frequency

domain notation, as in (10), and premultiplying \mathbf{Y} by \mathbf{Q}^* we can rewrite (23) as

$$\begin{aligned}
\tilde{\mathbf{Y}} &= \mathbf{Q}^* \mathbf{Y} = \mathbf{Q}^* \begin{bmatrix} \mathbf{Q} \boldsymbol{\Lambda}^{(1,1)} \mathbf{Q}^* & \mathbf{Q} \boldsymbol{\Lambda}^{(1,2)} \mathbf{Q}^* & \dots & \mathbf{Q} \boldsymbol{\Lambda}^{(1,M_t)} \mathbf{Q}^* \end{bmatrix} \begin{bmatrix} \check{\mathbf{X}}^{(1)} \\ \check{\mathbf{X}}^{(2)} \\ \vdots \\ \check{\mathbf{X}}^{(M_t)} \end{bmatrix} + \mathbf{Q}^* \mathbf{Z} \\
&= \begin{bmatrix} \boldsymbol{\Lambda}^{(1,1)} & \boldsymbol{\Lambda}^{(1,2)} & \dots & \boldsymbol{\Lambda}^{(1,M_t)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{Q}}^* \mathbf{X}^{(1)} \\ \tilde{\mathbf{Q}}^* \mathbf{X}^{(2)} \\ \vdots \\ \tilde{\mathbf{Q}}^* \mathbf{X}^{(M_t)} \end{bmatrix} + \mathbf{Q}^* \mathbf{Z} \\
&= \underbrace{\begin{bmatrix} \boldsymbol{\Lambda}^{(1,1)} & \boldsymbol{\Lambda}^{(1,2)} & \dots & \boldsymbol{\Lambda}^{(1,M_t)} \end{bmatrix}}_{\boldsymbol{\Lambda}} \begin{bmatrix} \tilde{\mathbf{Q}}^* & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{Q}}^* & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & \tilde{\mathbf{Q}}^* \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{X}^{(1)} \\ \mathbf{X}^{(2)} \\ \vdots \\ \mathbf{X}^{(M_t)} \end{bmatrix}}_{\mathbf{X}} + \underbrace{\mathbf{Q}^* \mathbf{Z}}_{\tilde{\mathbf{Z}}} \quad (24)
\end{aligned}$$

where $\tilde{\mathbf{Q}}^* \in \mathbb{C}^{T_s \times (T_s - \nu)}$ is a matrix obtained by deleting the last ν columns and $\tilde{\mathbf{Z}}$ still has i.i.d. Gaussian entries. Observe that since $\tilde{\mathbf{Q}}^*$ is a $T_s \times (T_s - \nu)$ Vandermonde matrix, it is a full rank matrix.

Let $\kappa \in \{1, \dots, M_t\}$ represent the antenna which has the maximum tap coefficient out of all the $M_T(\nu + 1)$ coefficients in the time domain, *i.e.*,

$$\max_{p \in \{1, \dots, M_t\}, l \in \{0, \dots, \nu\}} |h_l^{(1,p)}|^2 \leq \max \left(|h_0^{(1,\kappa)}|^2, \dots, |h_\nu^{(1,\kappa)}|^2 \right).$$

Define a selection matrix $\mathbf{S} \in \mathbb{C}^{(T_s-\nu) \times T_s}$ such that,

$$\begin{aligned} \mathbf{S}\Lambda \begin{bmatrix} \tilde{\mathbf{Q}}^* & 0 & \dots & 0 \\ 0 & \tilde{\mathbf{Q}}^* & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & \tilde{\mathbf{Q}}^* \end{bmatrix} &= \begin{bmatrix} \mathbf{S}\Lambda^{(1,1)}\tilde{\mathbf{Q}}^* & \mathbf{S}\Lambda^{(1,2)}\tilde{\mathbf{Q}}^* & \dots & \mathbf{S}\Lambda^{(1,M_t)}\tilde{\mathbf{Q}}^* \end{bmatrix} \\ \stackrel{(a)}{=} \begin{bmatrix} \hat{\Lambda}^{(1,1)}\hat{\mathbf{Q}}^* & \hat{\Lambda}^{(1,2)}\hat{\mathbf{Q}}^* & \dots & \hat{\Lambda}^{(1,M_t)}\hat{\mathbf{Q}}^* \end{bmatrix} &= \underbrace{\begin{bmatrix} \hat{\Lambda}^{(1,1)} & \hat{\Lambda}^{(1,2)} & \dots & \hat{\Lambda}^{(1,M_t)} \end{bmatrix}}_{\hat{\Lambda}} \underbrace{\begin{bmatrix} \hat{\mathbf{Q}}^* & 0 & \dots & 0 \\ 0 & \hat{\mathbf{Q}}^* & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & \hat{\mathbf{Q}}^* \end{bmatrix}}_{\check{\mathbf{Q}}^*} \end{aligned}$$

where, $\hat{\Lambda}^{(1,i)} \in \mathbb{C}^{(T_s-\nu) \times (T_s-\nu)}$ and in particular, $\hat{\Lambda}^{(1,\kappa)} = \text{diag}(\{\lambda_l^{(1,\kappa)} : l \in \mathcal{G}^{(1,\kappa)}\})$. The step (a) is valid above as $\hat{\Lambda}^{(1,i)}$ is a diagonal matrix. Therefore $\mathbf{S}\hat{\Lambda}^{(1,i)}$ will have exactly $T_s - \nu$ columns with non zero entries and will have ν columns with all zero entries. Therefore $\mathbf{S}\hat{\Lambda}^{(i)}\tilde{\mathbf{Q}}^*$ can be written as $\hat{\Lambda}^{(i)}\hat{\mathbf{Q}}^*$ where $\hat{\Lambda}^{(i)}$ is as defined above. Also $\hat{\mathbf{Q}}^* \in \mathbb{C}^{(T_s-\nu) \times (T_s-\nu)}$ is the matrix $\tilde{\mathbf{Q}}^*$ with the ν rows corresponding to $\{\lambda_l^{(1,\kappa)} : l \in \overline{\mathcal{G}^{(1,\kappa)}}\}$ deleted and $\check{\mathbf{Q}}^* \in \mathbb{C}^{(T_s-\nu)M_t \times (T_s-\nu)M_t}$. Using the same selection matrix for the whole block of T_b symbols we have,

$$\hat{\mathbf{Y}} = \mathbf{S}\tilde{\mathbf{Y}} = \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X} + \hat{\mathbf{Z}}, \quad (25)$$

where $\hat{\mathbf{Z}}$ is still iid Gaussian as it is obtained by deleting ν rows from $\tilde{\mathbf{Z}}$. Now we will impose constraints on the codewords \mathbf{X} to ensure the diversity embedding for MISO ISI channels.

A. Codebook Constraints

For transmission we consider a superposition of two $(T_s - \nu)M_t \times (T_s - \nu)M_t$ codebooks \mathcal{X}_H and \mathcal{X}_L of rates $(T_s - \nu)r_H$ and $(T_s - \nu)r_L$ respectively satisfying the following design criteria:

- 1) For $r_H \in [0, 1]$ and defining $\Delta\mathbf{X}_H = \mathbf{X}_H - \mathbf{X}'_H \neq 0$, for all $\mathbf{X}_H, \mathbf{X}'_H \in \mathcal{X}_H$ we require that,

$$\|\mathbf{X}_H\|_F^2 \leq (T_s - \nu)M_t \text{SNR} \leq (T_s - \nu)M_t T_s \text{SNR} \quad (26)$$

$$\min_{\Delta\mathbf{X}_H} \det(\Delta\mathbf{X}_H \Delta\mathbf{X}_H^*) \geq \text{SNR}^{(T_s-\nu)-(T_s-\nu)r_H} \quad (27)$$

2) For $r_L, \beta \in [0, 1]$ and defining $\Delta \mathbf{X}_L = \mathbf{X}_L - \mathbf{X}'_L \neq 0$, for all $\mathbf{X}_L, \mathbf{X}'_L \in \mathcal{X}_L$ we require that,

$$\|\mathbf{X}_L\|_F^2 \leq (T_s - \nu)M_t \text{SNR}^{1-\beta} \leq (T_s - \nu)M_t T_s \text{SNR}^{1-\beta} \quad (28)$$

$$\min_{\Delta \mathbf{X}_L} \det(\Delta \mathbf{X}_L \Delta \mathbf{X}_L^*) \geq \text{SNR}^{(T_s - \nu)M_t - (T_s - \nu)\beta - (T_s - \nu)r_L} \quad (29)$$

We will use superposition coding from $\mathcal{X}_H, \mathcal{X}_L$ so that $\mathbf{X} = \mathbf{X}_H + \mathbf{X}_L$. A particular set of codebooks satisfying these properties is constructed in [12] therefore establishing existence of codes with these properties. Since we are padding every $T_s - \nu$ symbols with ν zeros, if the code \mathcal{X}_H is designed with rate $(T_s - \nu)r_H$ the effective rate of communication is $\frac{(T_s - \nu)r_H}{T_s}$. Also, because of the energy constraint we have,

$$\|\Delta \mathbf{X}_H\|_F^2 = \text{tr}(\Delta \mathbf{X}_H \Delta \mathbf{X}_H^*) \leq \text{SNR}, \quad \|\Delta \mathbf{X}_L\|_F^2 = \text{tr}(\Delta \mathbf{X}_L \Delta \mathbf{X}_L^*) \leq \text{SNR}^{1-\beta} \quad (30)$$

B. Successive Refinement

Assuming transmission using the superposition coding as in Section IV-A, (25) is equivalent to,

$$\hat{\mathbf{Y}} = \hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{X}_H + \hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{X}_L + \hat{\mathbf{Z}} \quad (31)$$

For decoding the higher layer we treat the signal on the lower layer as noise. Representing $\|\cdot\|$ to be the Frobenius norm, the decoding rule for \mathbf{X}_H is given by,

$$\hat{\mathbf{X}}_H = \underset{\mathbf{X}_H}{\text{argmin}} \|\hat{\mathbf{Y}} - \hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{X}_H\|^2. \quad (32)$$

Using this decoding rule, the pairwise error probability can be upper bounded as in the following lemma.

Lemma 4.1: The pairwise error probability of detecting the sequence \mathbf{X}'_H given that \mathbf{X}_H was transmitted is upper bounded by,

$$P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) \leq Q \left(\|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\| - 2 \sum_{i=1}^{(T_s - \nu)M_t} \|\hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{x}_L^{(i)}\| \right), \quad (33)$$

where $\mathbf{x}_L^{(i)}$ is the i^{th} column of \mathbf{X}_L .

The proof of this lemma can be done using standard techniques and the details are in the appendix. Note that the error probability depends on the Frobenius norm of $(\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H))$, which is related to the singular values of $\check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)$ and $\hat{\Lambda}$. Therefore, we get bounds on the singular values of these two matrices in the Lemmas 4.2 and 4.3 and defer the proof to the appendix.

Lemma 4.2: Representing $\Delta\mathbf{X}_H = \mathbf{X}_H - \mathbf{X}'_H \neq 0$ for $\mathbf{X}_H, \mathbf{X}'_H \in \mathcal{X}_H$, $\check{\mathbf{Q}}^* \Delta\mathbf{X}_H \Delta\mathbf{X}_H^* \check{\mathbf{Q}}$ can be written as,

$$\check{\mathbf{Q}}^* \Delta\mathbf{X}_H \Delta\mathbf{X}_H^* \check{\mathbf{Q}} = \mathbf{R} \mathbf{D}_2^2 \mathbf{R}^*, \quad (34)$$

where \mathbf{R} is a unitary matrix chosen such that $\mathbf{D}_2^2 = \text{diag}(\xi_1^2, \xi_2^2, \dots, \xi_{(T_s-\nu)M_t}^2)$ and $\xi_1^2 \leq \xi_2^2 \leq \dots \leq \xi_{(T_s-\nu)M_t}^2$. Then we have the following bounds on ξ ,

$$\prod_{i=1}^{T_b} \xi_i^2 \geq \text{SNR}^{(T_s-\nu)M_t - (T_s-\nu)r} \quad (35)$$

$$\max_{i \in \{1, \dots, (T_s-\nu)M_t\}} (\xi_i^2) \leq \text{SNR}. \quad (36)$$

The decomposition of $\hat{\mathbf{\Lambda}} \hat{\mathbf{\Lambda}}^*$ can be used to get the representation of $\hat{\mathbf{\Lambda}}^* \hat{\mathbf{\Lambda}}$ as summarized in the following lemma.

Lemma 4.3: $\hat{\mathbf{\Lambda}}^* \hat{\mathbf{\Lambda}} \in \mathbb{C}^{T_b \times T_b}$ can be represented as $\hat{\mathbf{\Lambda}}^* \hat{\mathbf{\Lambda}} = \mathbf{V}^* \mathbf{D}_3^2 \mathbf{V}$, where $\mathbf{V} \in \mathbb{C}^{(T_s-\nu)M_t \times (T_s-\nu)M_t}$ is a unitary matrix chosen such that $\mathbf{D}_3^2 = \text{diag}(\gamma_1^2, \gamma_2^2, \dots, \gamma_{(T_s-\nu)}^2, \dots, \gamma_{(T_s-\nu)M_t}^2)$ and $\gamma_1^2 \geq \gamma_2^2 \dots \geq \gamma_{(T_s-\nu)M_t}^2$. Then,

$$\gamma_i^2 \doteq \max_{i \in \{0, 1, \dots, \nu\}} |h_i^{(1, \kappa)}|^2 = \lambda^2 \quad i \leq (T_s - \nu) \quad (37)$$

and $\gamma_i^2 = 0$ for $i > (T_s - \nu)$.

Combining these two lemmas and using the optimal decoder derived in Lemma 4.1 we can derive the following lemma on the exponential decay of error probability:

Lemma 4.4: Consider communication over a ν tap MISO ISI channel using codewords from \mathcal{X}_H and \mathcal{X}_L as described in section IV-A. For a finite period of communication (finite $T_s T_b$), given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$, the error probability of detecting the set of symbols sent from the higher constellation (\mathcal{X}_H) denoted by $P_e^H(\text{SNR})$ decays exponentially in SNR .

With these lemmas for the MISO channel, the successive refinement of the MISO channel can be stated as:

Theorem 4.5: The diversity multiplexing trade-off for a ν tap point to point MISO ISI channel is successively refinable, *i.e.*, for any multiplexing gains r_H and r_L such that $r_H + r_L \leq \frac{T_s - \nu}{T_s}$ the achievable

diversity orders given by $D_H(r_H)$ and $D_L(r_L)$ are bounded as,

$$(\nu + 1)M_t \left(1 - \frac{T_s}{(T_s - \nu)} r_H \right) \leq D_H(r_H) \leq (\nu + 1)M_t (1 - r_H), \quad (38)$$

$$(\nu + 1)M_t \left(1 - \frac{T_s}{(T_s - \nu)} (r_H + r_L) \right) \leq D_L(r_L) \leq (\nu + 1)M_t (1 - (r_H + r_L)) \quad (39)$$

where T_s is finite and does not grow with SNR.

Proof: To show the successive refinement we use superposition coding and assume two streams using codewords from \mathcal{X}_H and \mathcal{X}_L as described in section IV-A. The upper bound in both (38) and (39) is trivial and follows from the matched filter bound. We will investigate the lower bound in (38). Using the particular transmission scheme of superposition of codewords from \mathcal{X}_H and \mathcal{X}_L , given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$, we know from Lemma 4.4 that the error probability decays exponentially in SNR . Therefore,

$$\begin{aligned} P_e^H(SNR) &= P(\mathcal{M}(1 - r_H))P_e(SNR | \mathbf{h} \in \mathcal{M}(1 - r_H)) + P(\overline{\mathcal{M}(1 - r_H)})P_e(SNR | \mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}) \\ &\leq P(\mathcal{M}(1 - r_H)) + P(\overline{\mathcal{M}(1 - r_H)})P_e(SNR | \mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}) \\ &\doteq SNR^{-M_t(\nu+1)(1-\tilde{r}_H)} + (1 - SNR^{-M_t(\nu+1)(1-\tilde{r}_H)}) P_e(SNR | \mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}). \end{aligned} \quad (40)$$

For decoding the higher layer we treat the signal on the lower layer as noise. Given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$ and choosing $\beta > \tilde{r}_H$, we conclude from Lemma 4.4 that the second term in (40) decays exponentially in SNR . Therefore,

$$P_e^H(SNR) \leq SNR^{-M_t(\nu+1)(1-\tilde{r}_H)} \quad (41)$$

or equivalently, $M_t(\nu + 1) \left(1 - \frac{T_s}{(T_s - \nu)} r_H \right) \leq D_H(r_H)$ for communication at a rate of $r_H = \frac{(T_s - \nu)}{T_s} \tilde{r}_H$. Once we have decoded the upper layer we subtract its contribution from the lower layer. Using the set $\mathcal{M}(1 - \tilde{r}_L - \beta)$, and by taking β arbitrarily close to \tilde{r}_H , we can conclude (39). Therefore, from (38) and (39) it follows that the diversity multiplexing trade-off for ISI channel with single degree of freedom is successively refinable. ■

V. DISCUSSION

The constraints on the codebook in Section IV-A specializes to simpler cases for particular channels. Thus, the inequalities in (27) and (29) are sufficient but not necessary conditions for successive refinability. For example, the coding scheme to achieve the successive refinement of the D-M trade-off in [5] for

transmission over flat fading channel is a special case of the codebook in Section IV-A with $\nu = 0$ and $T_s = 1$. Similarly the uncoded QAM constellations used for the SISO/SIMO ISI channel in [9] can be shown to satisfy the constraints in equations (27) and (29) with $T_b = 1$.

Theorem 4.5 implies that for ISI fading channels with a single degree of freedom, it is possible to design (asymptotically in SNR) ideal *opportunistic* codes. The existence of (almost) ideal opportunistic codes is surprising since one would have expected the behavior for the ISI channel to be closer to the flat-fading multiple-degrees-of-freedom case, where the D-M trade-off was not successively refinable [4].

We can also interpret the successive refinability by using the rate region for the broadcast channel with user channels corresponding to the typical error events of the corresponding diversity levels as shown in Figure 4. It demonstrates that as SNR grows the shape of the Gaussian broadcast capacity region becomes closer to a trapezoid. This implies that by reducing the rate slightly for the high priority user (worse channel) we can significantly increase the rate for the low priority user (better channel). The figure is plotted for different SNR levels, and the result shows that asymptotically the loss in rate for this exchange becomes very small. This gives an engineering interpretation of the successive refinement result.

This paper demonstrated the successive refinement of diversity for ISI fading channels with a single degree of freedom. However, many questions remain open. Given the result of [4] for flat fading channels, it is natural to expect that the successive refinement property will not hold for MIMO ISI fading channels. However, there is an advantage of layering information, and characterizing the rate-diversity tuples for MIMO channels would be an important open question. Other issues including practical decoding schemes and impact of this on multimedia applications would be natural avenues of future enquiry.

APPENDIX

A. Proof of Lemma 4.1

Proof: Representing $\|\cdot\|$ to be the Frobenius norm and using the decoding rule in (32), we get,

$$\begin{aligned} P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) &= Pr \left(\|\hat{\mathbf{Y}} - \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_H\|^2 > \|\hat{\mathbf{Y}} - \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}'_H\|^2 \right) \\ &= Pr \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_L + \hat{\mathbf{Z}}\|^2 > \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_H - \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}'_H + \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_L + \hat{\mathbf{Z}}\|^2 \right). \end{aligned} \quad (42)$$

Denote $\mathbf{x}_L^{(i)}, \mathbf{z}^{(i)}, \mathbf{x}_H^{(i)}, \mathbf{y}^{(i)}$ to be the i^{th} columns of $\mathbf{X}_L, \hat{\mathbf{Z}}, \mathbf{X}_H$ and $\hat{\mathbf{Y}}$ respectively. With these definitions we can expand the left hand side and the right hand side of the inequality above as,

$$\begin{aligned} LHS &= \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_L + \hat{\mathbf{Z}}\|^2 = \sum_{i=1}^{T_b} \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} + \mathbf{z}^{(i)}\|^2 \quad \text{and} \\ RHS &= \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_H - \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}'_H + \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{X}_L + \hat{\mathbf{Z}}\|^2 = \sum_{i=1}^{T_b} \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_H^{(i)} - \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_H^{(i')} + \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} + \mathbf{z}^{(i)}\|^2 \\ &= \sum_{i=1}^{T_b} \left\{ \|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 + \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} + \mathbf{z}^{(i)}\|^2 + 2\text{Re} \left\langle \hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')}), \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} + \mathbf{z}^{(i)} \right\rangle \right\}. \end{aligned}$$

Substituting these expansions in (42) and expanding we get,

$$\begin{aligned} P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) &= Pr \left(- \sum_{i=1}^{T_b} \left(2\text{Re} \left\langle \hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')}), \mathbf{z}^{(i)} \right\rangle \right) > \sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 \right) + \right. \\ &\quad \left. \sum_{i=1}^{T_b} \left(2\text{Re} \left\langle \hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')}), \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} \right\rangle \right) \right). \end{aligned}$$

Defining,

$$\mathbf{u}^{(i)} = \frac{\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})}{\sqrt{\sum_{i=1}^{T_b} \|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2}}$$

we can see that, $\sum_{i=1}^{T_b} \mathbf{u}^{(i)*}\mathbf{u}^{(i)} = \sum_{i=1}^{T_b} \|\mathbf{u}^{(i)}\|^2 = 1$. Dividing both sides by $\sqrt{\sum_{i=1}^{T_b} \|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2}$

we get,

$$P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) = Pr \left(v > \sqrt{\sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 \right)} + \sum_{i=1}^{T_b} \left(2\text{Re} \left\langle \mathbf{u}^{(i)}, \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} \right\rangle \right) \right)$$

where,

$$v = \sum_{i=1}^{T_b} \left(2\text{Re} \left\langle \mathbf{u}^{(i)}, -\mathbf{z}^{(i)} \right\rangle \right) = \mathcal{CN} \left(0, \mathbb{E} \left(\sum_{i=1}^{T_b} \mathbf{u}^{(i)*}\mathbf{u}^{(i)} \right) \right) = \mathcal{CN} (0, 1).$$

Therefore,

$$\begin{aligned} P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) &= Q \left(\sqrt{\sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 \right)} + \sum_{i=1}^{T_b} \left(2\text{Re} \left\langle \mathbf{u}^{(i)}, \hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)} \right\rangle \right) \right) \\ &\leq Q \left(\sqrt{\sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 \right)} - 2 \sum_{i=1}^{T_b} \left(\|\mathbf{u}^{(i)}\| \cdot \|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)}\| \right) \right) \\ &\leq Q \left(\sqrt{\sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*(\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i')})\|^2 \right)} - 2 \sum_{i=1}^{T_b} \left(\|\hat{\Lambda}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)}\| \right) \right). \end{aligned}$$

Since $\sum_{i=1}^{T_b} \|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{x}_H^{(i)} - \mathbf{x}_H^{(i)'})\|^2 = \|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\|^2$ we can rewrite the equation to get the desired result *i.e.*,

$$P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{h}, \mathbf{X}_L) \leq Q \left(\|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\| - 2 \sum_{i=1}^{T_b} (\|\hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{x}_L^{(i)}\|) \right).$$

■

B. Proof of Lemma 4.2

Proof: Since $\check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^* \check{\mathbf{Q}}$ is a Hermetian matrix it can be written as in (34). Since $\hat{\mathbf{Q}}^*$ is still a full rank Vandermonde matrix which does not depend on SNR it follows that is a full rank matrix independent of SNR . Since determinant of a matrix is product of its eigenvalues we get,

$$\begin{aligned} \prod_{i=1}^{(T_s-\nu)M_t} \xi_i^2 &= \det(\check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^* \check{\mathbf{Q}}) = \det(\check{\mathbf{Q}} \check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^*) = \det(\check{\mathbf{Q}} \check{\mathbf{Q}}^*) \det(\Delta \mathbf{X}_H \Delta \mathbf{X}_H^*) \\ &\doteq \det(\Delta \mathbf{X}_H \Delta \mathbf{X}_H^*) \geq SNR^{(T_s-\nu)M_t - (T_s-\nu)r} \quad \text{from (27)}. \end{aligned}$$

Combining submultiplicativity of the Frobenius norm² with the fact that the sum of the eigenvalues is equal to the trace, for all $i \in \{1, \dots, (T_s - \nu)M_t\}$ we get,

$$\xi_i^2 \leq \sum_{i=1}^{(T_s-\nu)M_t} \xi_i^2 = \text{tr} \left(\check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^* \check{\mathbf{Q}} \right) = \|\check{\mathbf{Q}}^* \Delta \mathbf{X}_H\|_F^2 \leq \|\check{\mathbf{Q}}^*\|_F^2 \|\Delta \mathbf{X}_H\|_F^2 \doteq SNR \quad \text{from (30)}.$$

■

C. Proof of Lemma 4.3

Proof: Observe that because of the way we have chosen our selection matrix,

$$\hat{\Lambda} \hat{\Lambda}^* = \text{diag}_{i \in \mathcal{G}(1, \kappa)} \left(\left\{ \sum_{p=1}^{M_t} |\lambda_i^{(1,p)}|^2 \right\} \right) \doteq \text{diag}_{i \in \mathcal{G}(1, \kappa)} \left(\left\{ |\lambda_i^{(1, \kappa)}|^2 \right\} \right)$$

as $|\lambda_i^{(1, \kappa)}|^2$ is the dominant term in the summation. From Lemma 3.1 we have that for $i \in \mathcal{G}(1, \kappa)$, $|\lambda_i^{(\kappa)}|^2 \doteq \max_{i \in \{0, 1, \dots, \nu\}} |h_i^{(1, \kappa)}|^2$. We know that the eigenvalues of $\hat{\Lambda} \hat{\Lambda}^*$ and $\hat{\Lambda}^* \hat{\Lambda}$ are identical with the remaining eigenvalues being equal to zero. Assume that \mathbf{D}_3^2 is represented as, $\mathbf{D}_3^2 = \text{diag} \left(\gamma_1^2, \gamma_2^2, \dots, \gamma_{(T_s-\nu)}^2, \dots, \gamma_{(T_s-\nu)M_t}^2 \right)$ where $\gamma_1^2 \geq \gamma_2^2 \dots \geq \gamma_{(T_s-\nu)M_t}^2$. The result then follows directly. ■

² $\|AB\|_F \leq \|A\|_F \|B\|_F$

D. Proof of Lemma 4.4

Proof: For a uniform choice of codewords for the lower layer from lemma 4.1, using the decoding rule in (32), the pairwise error probability of detecting the sequence \mathbf{X}'_H given that \mathbf{X}_H was transmitted is upper bounded by,

$$\begin{aligned} P_e(\mathbf{X} \rightarrow \mathbf{X}' | \mathbf{h} \in \overline{\mathcal{M}(1-r_H)}) &= \sum_{\mathbf{X}_L} Pr(\mathbf{X}_L) P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{X}_L, \mathbf{h} \in \overline{\mathcal{M}(1-r_H)}) \\ &= SNR^{-r_L T} \sum_{\mathbf{X}_L} P_e(\mathbf{X}_H \rightarrow \mathbf{X}'_H | \mathbf{X}_L, \mathbf{h} \in \overline{\mathcal{M}(1-r_H)}) \leq Q \left(\underbrace{\|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\| - 2 \sum_{i=1}^{T_b} (\|\hat{\Lambda} \check{\mathbf{Q}}^* \mathbf{x}_L^{(i)}\|)}_{\Omega} \right) \end{aligned}$$

where $\mathbf{x}_L^{(i)}$ is the i^{th} column of \mathbf{X}_L . We will now get a lower bound on Ω in the equation above to get an upper bound to the error probability. Using $tr(AB) = tr(BA)$ and representing $\Delta \mathbf{X}_H = \mathbf{X}_H - \mathbf{X}'_H$, for the first term in Ω we get that,

$$\|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\|^2 = tr \left(\hat{\Lambda} \check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^* \check{\mathbf{Q}} \hat{\Lambda}^* \right) = tr \left(\check{\mathbf{Q}}^* \Delta \mathbf{X}_H \Delta \mathbf{X}_H^* \check{\mathbf{Q}} \hat{\Lambda}^* \hat{\Lambda} \right) \quad (43)$$

Substituting the SVD from (34) and Lemma 4.3 into (43) we get,

$$\begin{aligned} \|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\|^2 &= tr \left(\mathbf{R} \mathbf{D}_2^2 \mathbf{R}^* \mathbf{V}^* \mathbf{D}_3^2 \mathbf{V} \right) = tr \left(\mathbf{V} \mathbf{R} \mathbf{D}_2^2 \mathbf{R}^* \mathbf{V}^* \mathbf{D}_3^2 \right) = tr \left(\mathbf{T} \mathbf{D}_2^2 \mathbf{T}^* \mathbf{D}_3^2 \right) \\ &= \sum_{i,j=1}^{(T_s-\nu)M_t} \gamma_i^2 \xi_j^2 |t_{ij}|^2 \end{aligned} \quad (44)$$

where $\mathbf{T} = \mathbf{V} \mathbf{R}$ is also an unitary matrix and t_{ij} is the (i, j) element of \mathbf{T} . Since,

$$\xi_1 \leq \xi_2 \leq \dots \leq \xi_{(T_s-\nu)M_t}, \quad \gamma_1^2 \geq \gamma_2^2 \dots \geq \gamma_{(T_s-\nu)M_t}^2$$

using similar reasoning as [14], [12] in (44) we get,

$$\begin{aligned} \|\hat{\Lambda} \check{\mathbf{Q}}^* (\mathbf{X}_H - \mathbf{X}'_H)\|^2 &\geq \sum_{i=1}^{(T_s-\nu)M_t} \gamma_i^2 \xi_i^2 \stackrel{(a)}{=} \sum_{i=1}^{(T_s-\nu)} \gamma_i^2 \xi_i^2 \stackrel{(b)}{\doteq} \sum_{i=1}^{(T_s-\nu)} \lambda^2 \xi_i^2 \geq \lambda^2 (T_s - \nu) \left[\prod_{i=1}^{(T_s-\nu)} \xi_i^2 \right]^{\frac{1}{(T_s-\nu)}} \\ &\stackrel{(c)}{\geq} \lambda^2 \left[\frac{SNR^{(T_s-\nu)M_t - (T_s-\nu)r_H}}{\prod_{i=(T_s-\nu)+1}^{(T_s-\nu)M_t} \xi_i^2} \right]^{\frac{1}{(T_s-\nu)}} \geq \lambda^2 \left[\frac{SNR^{(T_s-\nu)M_t - (T_s-\nu)r_H}}{\prod_{i=(T_s-\nu)+1}^{(T_s-\nu)M_t} \xi_{max}^2} \right]^{\frac{1}{(T_s-\nu)}} \\ &\stackrel{(d)}{\geq} \lambda^2 \left[\frac{SNR^{(T_s-\nu)M_t - (T_s-\nu)r_H}}{SNR^{(T_s-\nu)M_t - (T_s-\nu)}} \right]^{\frac{1}{(T_s-\nu)}} = \lambda^2 [SNR^{(T_s-\nu)(1-r_H)}]^{\frac{1}{(T_s-\nu)}} = \lambda^2 SNR^{(1-r_H)}. \end{aligned}$$

where (a) follows from (37), (b) follows from AM \geq GM, (c) follows from (35) and (d) is from (36). Given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$ we can write,

$$\lambda^2 \doteq |h_l^{(1,\kappa)}|^2 \doteq \text{SNR}^{2\epsilon} \text{SNR}^{-(1-r_H)} \quad (45)$$

where $\epsilon > 0$. Therefore,

$$\|\hat{\mathbf{A}}\check{\mathbf{Q}}^*(\mathbf{X}_H - \mathbf{X}'_H)\|^2 \geq \text{SNR}^{2\epsilon} \text{SNR}^{-(1-r_H)} \text{SNR}^{(1-r_H)} = \text{SNR}^{2\epsilon}. \quad (46)$$

For the second term in Ω , from the submultiplicativity of the Frobenius Norm we get

$$\begin{aligned} \|\hat{\mathbf{A}}\check{\mathbf{Q}}^*\mathbf{x}_L^{(i)}\|^2 &\leq \|\hat{\mathbf{A}}\|^2 \|\check{\mathbf{Q}}^*\|^2 \|\mathbf{x}_L^{(i)}\|^2 \doteq \|\hat{\mathbf{A}}\|^2 \|\mathbf{x}_L^{(i)}\|^2 \doteq \lambda^2 \|\mathbf{x}_L^{(i)}\|^2 \leq \lambda^2 \|\mathbf{X}_L\|^2 \\ &\stackrel{(a)}{\leq} \text{SNR}^{2\epsilon} \text{SNR}^{-(1-r_H)} \text{SNR}^{1-\beta} \doteq \text{SNR}^{2\epsilon} \text{SNR}^{r_H-\beta} \end{aligned} \quad (47)$$

where (a) follows from (28) and (45). Therefore, combining equations (46) and (47) we can lower bound Ω as,

$$\Omega \geq \text{SNR}^\epsilon - T_b \text{SNR}^{\epsilon + \frac{(r_H-\beta)}{2}} \doteq \text{SNR}^\epsilon \left(1 - T_b \text{SNR}^{\frac{(r_H-\beta)}{2}}\right) \doteq \text{SNR}^\epsilon,$$

where the last step is valid as $\beta > r_H$. Therefore,

$$P_e^H(\mathbf{X}_H \rightarrow \mathbf{X}'_H) \leq Q(\text{SNR}^\epsilon). \quad (48)$$

Note that $Q(x)$ decays exponentially in x for large x i.e., $Q(x) \leq e^{-\frac{x^2}{2}}$. By the union bound it then follows that given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$, $P_e^H(\text{SNR})$ decays exponentially in SNR. From the union bound and the exponential decay of $Q(x)$ it then follows that given that $\mathbf{h} \in \overline{\mathcal{M}(1 - r_H)}$, $P_e^H(\text{SNR})$ decays exponentially in SNR. \blacksquare

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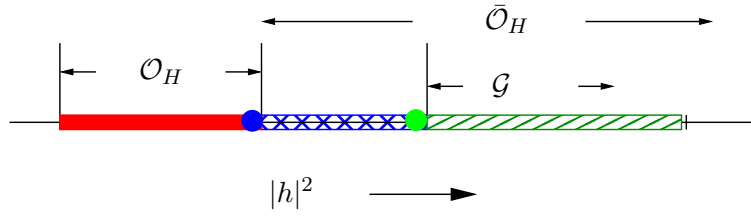


Fig. 1. Outage events in the classical setting and for diversity embedded coding

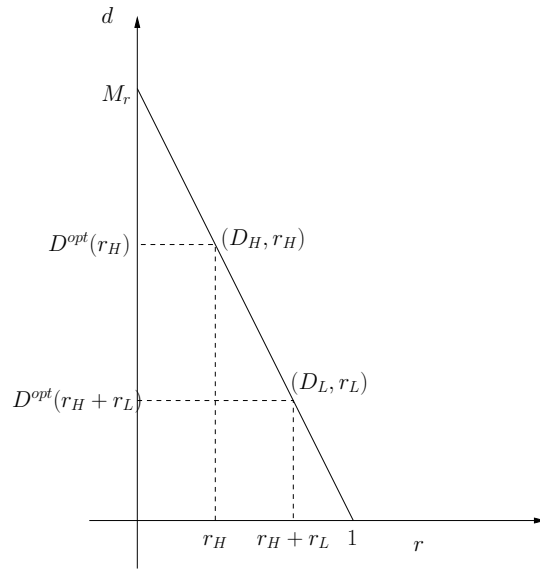


Fig. 2. Successive refinement for a flat fading channel with M_r receive antennas and one transmit antenna.

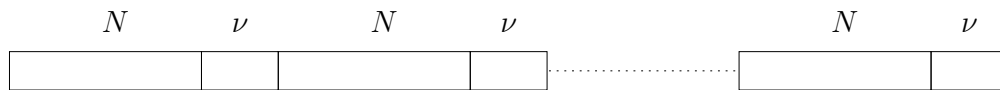


Fig. 3. Coding Scheme for MISO channels

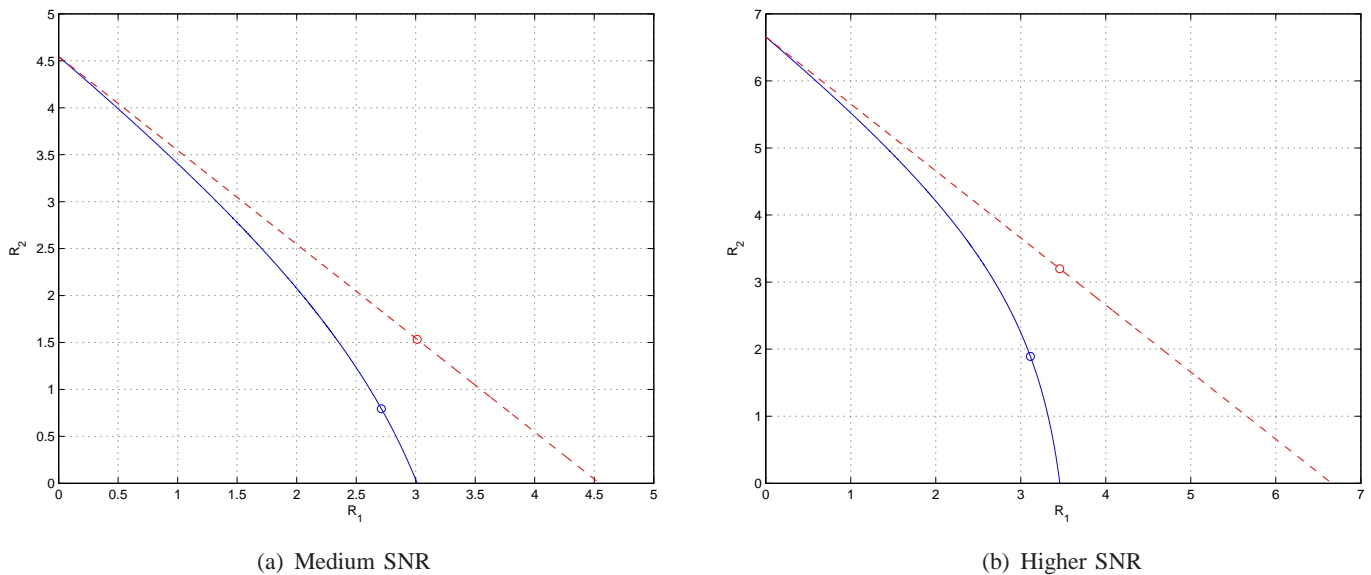


Fig. 4. The rate region illustrated for the scalar Gaussian broadcast channel with the rate for the weaker channel on the x-axis and the stronger channel on the y-axis. The rates are illustrated for $M_t = 2, M_r = 1$, and channels for given outage probabilities. The rates in Figure (a) are for 20 dB SNR, and typical channels corresponding to outage of $p_H = 10^{-2}, p_L = 10^{-1}$. The rates in (b) are for a higher SNR of 30 dB, and we notice that the region looks closer to a trapezoid (*i.e.*, the curve hugs the 45 degree line shown for illustration, and departs almost vertically downwards). This shows that for a small reduction in the rate for the worse channel, a large increase for the better channel can be obtained. Asymptotically this trapezoidal shape gives the intuition for the successive refinement of diversity property since the reduction needed for the worse channel is small (in terms of multiplexing rate) and still attaining the optimal sum rate.