

On the Oblivious Transfer Capacity of the Degraded Wiretapped Binary Erasure Channel

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Abstract—We study oblivious transfer (OT) between Alice and Bob in the presence of an eavesdropper Eve over a degraded wiretapped binary erasure channel from Alice to Bob and Eve. In addition to the privacy goals of oblivious transfer between Alice and Bob, we require privacy of Alice and Bob’s private data from Eve. In previous work we derived the OT capacity (in the honest-but-curious model) of the wiretapped binary independent erasure channel where the erasure processes of Bob and Eve are independent. Here we derive a lower bound on the OT capacity in the same secrecy model when the wiretapped binary erasure channel is degraded in favour of Bob.

I. INTRODUCTION

In secure multiparty computation, mutually distrusting users want to collaborate in computing functions of their data. They want to do this in such a way that no user derives additional information about other users’ data than the function they compute. This has applications in several areas including data-mining, voting, auctions etc. [5]. In general, secure computation is not possible between users who only have access to private/common randomness and noiseless communication [9]. For two-user secure computation, a noisy channel between the users (in addition to a noise-free public channel) provides a stochastic resource on which secure computation can be based [6]. Oblivious transfer (OT), which is a specific two-user secure computation, has been proposed as a primitive on which all secure computation can be based [7], [8], and OT itself can be obtained from noisy channels.

In 1-of-2 string OT, one of the users, say, Alice, has two bit-strings of equal length. The other user, say, Bob, wants to learn exactly one of the two strings. Bob does not want Alice to find out which of the two strings he wants, while Alice wants to ensure that Bob does not learn anything more than one of the strings. The OT capacity of a discrete memoryless channel is the largest rate of the string-length per channel use that can be achieved. We assume that the users are honest-but-curious, that is, they follow the protocol agreed upon but they may try to gain illegitimate information about the other user’s private data from everything they have learned at the end of the protocol. For such users, Nascimento and Winter [13] obtained a lower bound on the OT capacity of noisy channels and distributed sources. Ahlswede and Csiszár [2] obtained lower bounds on the honest-but-curious OT capacity for generalized erasure channels. These lower bounds are tight when the erasure probability is at least $\frac{1}{2}$. Pinto et. al. [14] showed that for erasure probability at least $\frac{1}{2}$, the OT capacity of generalized

erasure channels remains the same even when the users are *malicious*, that is, even if a dishonest user arbitrarily deviates from the protocol.

Presence of third parties is a natural concern when using noisy channels. Motivated by this, OT over wiretapped binary erasure channel was studied in [11]. Building on the ideas from [2], [13], the OT capacity of this channel was characterized there for the honest-but-curious model. Both 2-privacy, where the eavesdropper may collude with either Alice or Bob, and 1-privacy, where there are no collusions, were considered. In [12], the problem of performing independent OTs between Alice and each of the other parties over a binary erasure broadcast channel was considered. Inner and outer bounds on the OT capacity region were presented which meet except in one regime of parameter ranges.

Here we study the OT capacity of the *degraded* wiretapped binary erasure channel. The problem presents some interesting new features. Oblivious transfer relies on the noise in the legitimate channel (Alice-to-Bob channel) to hide information of Alice and Bob from each other. In a degraded erasure channel, the wiretapper obtains more information about the noise process on the legitimate channel (compared to an independent erasure channel where it receives no information on this noise process from the channel). Our achievable scheme is, in fact, more involved compared to the one in [11], [12] precisely because of this fact. This is in contrast with secret key agreement using broadcast channels with public discussion, where optimal schemes are simpler for the degraded channel in the sense that no public discussion is needed to achieve secret key capacity [1], [10]. While we do not prove the optimality of our scheme, we believe that the additional complexity is unavoidable.

In Section II, we present the formal problem definition. The main result is presented in Section III, and the proof is presented in Section IV.

II. PROBLEM STATEMENT

In the setup of Fig. 1, Alice is connected to Bob and an eavesdropper Eve over a broadcast channel $p_{YZ|X}$. Additionally, there is a public channel of unlimited capacity, over which Alice and Bob can take turns to send messages. Each message sent over this public channel is received by all users. Alice’s private data is a pair of strings K_0, K_1 which are m -bit each, while Bob’s private data is a choice bit U . K_0, K_1, U are

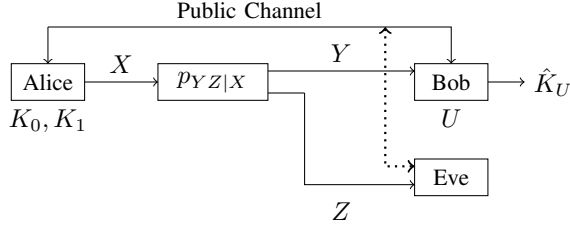


Fig. 1: Setup for oblivious capacity over a broadcast channel

independent and uniform over their respective alphabets. The goal is for Alice and Bob to do an OT using (K_0, K_1, U) , without Eve learning anything about (K_0, K_1, U) .

We consider the degraded broadcast channel setup shown in Fig. 2. This is a special case of Fig. 1. Here the broadcast channel $p_{YZ|X}$ is made up of a cascade of two independent binary erasure channels (BECs), $\text{BEC}(\epsilon_1)$ with erasure probability ϵ_1 followed by a $\text{BEC}(\epsilon_2)$. That is, $X \in \{0, 1\}, Y, Z \in \{0, 1, e\}$, and $p_{YZ|X} = p_{Y|X}p_{Z|Y}$ with $p_{Y|X}(e|1) = p_{Y|X}(e|0) = \epsilon_1$, $p_{Y|X}(1|1) = p_{Y|X}(0|0) = 1 - \epsilon_1$, and $p_{Z|Y}(e|e) = 1, p_{Z|Y}(e|1) = p_{Z|Y}(e|0) = \epsilon_2$, $p_{Z|Y}(1|1) = p_{Z|Y}(0|0) = 1 - \epsilon_2$.

Definition 1. Let n, m be positive integers. An (n, m) -protocol is an exchange of messages over the setup of Fig. 1. Alice transmits a bit over the broadcast channel $p_{YZ|X}$ at each time instant $t = 1, 2, \dots, n$. Alice's private strings are m -bits each. Before each channel use by Alice and also after Alice's last channel use, Alice and Bob can take turns to send an arbitrary but finite number of messages over the public channel.

We denote by \mathbf{F} the transcript of the public channel at the end of the protocol.

Definition 2. The final view of a user is the set of random variables it generates and receives over the execution of the protocol. The final views of Alice, Bob and Eve are, respectively,

$$\begin{aligned} V_A &:= (K_0, K_1, X^n, \mathbf{F}), \\ V_B &:= (U, Y^n, \mathbf{F}), \\ V_E &:= (Z^n, \mathbf{F}). \end{aligned}$$

At the end of the protocol, Bob generates an estimate \hat{K}_U of K_U , as a function of its final view V_B .

Definition 3. The rate r_n of an (n, m) -protocol is

$$r_n := \frac{m}{n} \quad (1)$$

Definition 4. A rate R is achievable in the setup of Fig. 1 if there exists a sequence of (n, m) -protocols such that, as $n \rightarrow \infty$, $r_n \rightarrow R$ and

$$P[\hat{K}_U \neq K_U] \rightarrow 0, \quad (2)$$

$$I(K_{\bar{U}}; V_B) \rightarrow 0, \quad (3)$$

$$I(U; V_A) \rightarrow 0, \quad (4)$$

$$I(K_0, K_1, U; V_E) \rightarrow 0. \quad (5)$$

Definition 5. The capacity C in the setup of Fig. 1 is

$$C := \sup\{R : R \text{ is achievable}\} \quad (6)$$

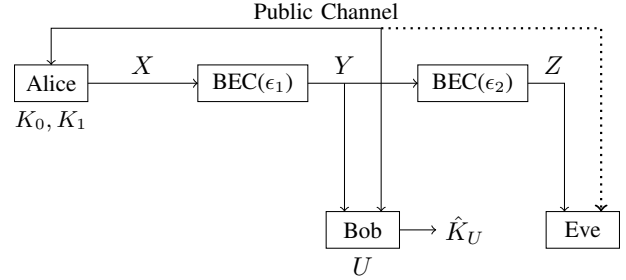


Fig. 2: Setup for oblivious transfer over a degraded binary erasure broadcast channel

III. SUMMARY OF RESULTS

Our main result is a lower bound on C , for the setup of Fig. 2.

Theorem 1.

$$\min\left\{\frac{1}{3}\epsilon_2(1 - \epsilon_1), \epsilon_1\right\} \leq C \leq \min\{\epsilon_2(1 - \epsilon_1), \epsilon_1\}.$$

Hence, when $\epsilon_1 \leq \frac{1}{3}\epsilon_2(1 - \epsilon_1)$, we have $C = \epsilon_1$. Note that the upper bound of $\epsilon_2(1 - \epsilon_1)$ in Theorem 1 follows from the fact that OT capacity is upper bounded by the secret key capacity of the wiretapped channel. This is because if Bob runs the protocol with the choice bit set deterministically to, say, 0, then K_0 is a secret key between Alice and Bob. The upper bound follows from the fact that $\epsilon_2(1 - \epsilon_1)$ is the secret key capacity of this wiretapped channel with public discussion [1], [10]. The upper bound of ϵ_1 follows from fact that this is an upper bound for two-party OT capacity of the binary erasure channel with erasure probability ϵ_1 [2].

Section IV gives a protocol that is used to prove the lower bound part of Theorem 1.

IV. ACHIEVABILITY OF THEOREM 1

In this section, we will describe a protocol which will be used to show that the lower bound in Theorem 1 is an achievable rate. We begin by presenting the main ideas used in this protocol, before presenting a formal description for it.

Let r be any rate smaller than $\min\{\frac{1}{3}\epsilon_2(1 - \epsilon_1), \epsilon_1\}$. In the protocol, Alice begins by transmitting a sequence X^n of independent bits, each equally likely to be 0 or 1. Bob receives the corresponding erased version Y^n , while Eve gets Z^n which is an erased version of Y^n (see Fig. 2). Let E denote the set of indices at which Y^n has erasures, and let \bar{E} be its complement. Out of the set \bar{E} , Bob picks a good set G of size $|G| = nr$ uniformly at random. Similarly, Bob also picks a bad set B of size $|B| = nr$ uniformly at random out of set E . Let $\tilde{G} := \bar{E} \setminus G$ denote the unerased indices which are not in G . Note that, $|\tilde{G}|$ is approximately $n(1 - \epsilon_1 - r)$. Similarly, $\hat{B} := E \setminus B$ is the set of erased indices which are not in B ,

and $|\tilde{B}|$ is approximately $n(\epsilon_1 - r)$. If $U = 0$, Bob assigns sets $(L_0, L_1) = (G, B)$, otherwise Bob sets $(L_0, L_1) = (B, G)$.

Bob first declares the sets \tilde{G}, \tilde{B} over the public channel. Thereafter, Bob forms sequential, disjoint subsets $(\tilde{G}_L, \tilde{G}_S)$ of the set \tilde{G} , where $|\tilde{G}_L|$ is about $\frac{2nr}{\epsilon_2}$ and $|\tilde{G}_S|$ is about $\frac{nr(1-\epsilon_2)}{\epsilon_2}$. Both \tilde{G}_L and \tilde{G}_S will be used to generate secret keys known to both Alice and Bob, but hidden from Eve. Note that for $r < \min\{\frac{1}{3}\epsilon_2(1-\epsilon_1), \epsilon_1\}$, all these sets of the required sizes can be formed.

Alice and Bob use the set of transmissions $X^n|_{\tilde{G}_L}$ to agree on a secret key S_L , secret from Eve. The size of S_L is $2nr$ bits. Similarly, Alice and Bob use $X^n|_{L_0 \cup \tilde{G}_S}$ to form a secret key S_0 and $X^n|_{L_1 \cup \tilde{G}_S}$ to form a secret key S_1 . Here, $|S_0| = |S_1| = nr$.

Bob now needs to reveal (L_0, L_1) to Alice, while hiding both these sets from Eve, so as not to reveal U to Eve. Towards this goal, Bob forms a set L which is an ordered version of the set $L_0 \cup L_1$. Then, Bob forms a binary vector Q of length $2nr$ as follows. For all $i = 0, 1, \dots, 2nr - 1$, the i th element of Q , denoted by Q_i will indicate whether the i th element of L , denoted by L_i , belongs to L_0 or L_1 . That is, $Q_i = 0$ when $L_i \in L_0$, otherwise $Q_i = 1$. Bob now sends $S_L \oplus Q$ to Alice, which is sufficient for Alice to recover (L_0, L_1) .

Alice finally sends the encrypted strings $K_0 \oplus S_0$ and $K_1 \oplus S_1$ to Bob over the public channel. Since Bob knows S_U completely (since he knows $X^n|_{L_U \cup \tilde{G}_S}$), Bob can recover K_U .

Before presenting the protocol more formally, we point out a comparison with the independent erasure case of [11]. In the independent erasure channel, since Eve has no side information about the erasure pattern of Bob, there is no need for Bob to encrypt (L_0, L_1) before sending it over the public channel. The additional burden of encrypting L_0, L_1 here requires Alice and Bob to generate a secret key S_L which is twice as long as each string K . Along with S_U , effectively, Alice and Bob agree on secret keys which are together thrice as long as each string K . This explains the $\frac{1}{3}$ factor in the rate achieved.

Protocol 1. Let $\delta \in (0, 1)$. Let $\tilde{\epsilon}_2 = \epsilon_2(1 - \delta)$. Let $r = \min\{\frac{1}{3}\tilde{\epsilon}_2(1 - \epsilon_1) - \theta_\delta, \epsilon_1 - \delta\}$ be the rate to be achieved, where $\theta_\delta = \delta(1 + \frac{2}{\tilde{\epsilon}_2})$.

Alice Transmits a sequence X^n of i.i.d. $\text{Ber}(\frac{1}{2})$ bits over the broadcast channel in Fig. 2.

Bob Receives the sequence Y^n from the output of $\text{BEC}(\epsilon_1)$ and forms the erased and unerased sets of indices of Y^n as, respectively,

$$E := \{i \in \{1, 2, \dots, n\} : Y_i = e\}$$

$$\bar{E} := \{i \in \{1, 2, \dots, n\} : Y_i \neq e\}$$

If $|\bar{E}| < n(1 - \epsilon_1 - \delta)$ or $|E| < n(\epsilon_1 - \delta)$, Bob declares an error and quits. Otherwise, Bob proceeds and forms the following sets out of E and \bar{E} :

$$G := \text{Unif}\{A \subset \bar{E} : |A| = n(r + \delta)\}$$

$$B := \text{Unif}\{A \subset E : |A| = n(r + \delta)\}$$

$$\tilde{G} := \bar{E} \setminus G$$

$$\tilde{B} := E \setminus B$$

where $\text{Unif}\{\cdot\}$ denotes a random, uniformly distributed choice over the collection of sets. Note that $|\tilde{G}| \geq n(1 - \epsilon_1 - r - \delta)$ and $|\tilde{B}| \geq n(\epsilon_1 - r - \delta)$. Bob reveals the sets \tilde{G}, \tilde{B} over the public channel. Bob now forms the sets L_0, L_1 as follows:

$$U = 0 : \quad L_0 = G, \quad L_1 = B$$

$$U = 1 : \quad L_0 = B, \quad L_1 = G$$

Let L be an ordered version of the set $L_0 \cup L_1$ and let $L_i, i = 0, 1, \dots, 2n(r + \delta) - 1$, be its i^{th} element. Bob forms a binary vector Q of $2n(r + \delta)$ bits, with elements labelled $Q_i, i = 0, 1, \dots, 2n(r + \delta) - 1$ defined as:

$$Q_i = \begin{cases} 0, & L_i \in L_0 \\ 1, & L_i \in L_1 \end{cases}$$

Bob takes the first $\frac{2n(r + \delta)}{\epsilon_2}$ indices in \tilde{G} and calls it set \tilde{G}_L . Bob also takes the next $\frac{nr(1 - \tilde{\epsilon}_2)}{\epsilon_2}$ indices in \tilde{G} and calls it set \tilde{G}_S . Bob then forms a secret key S_L using $X^n|_{\tilde{G}_L}$, where S_L is known to Alice but hidden from Eve. Here, S_L is $2n(r + \delta)$ bits long. Finally, Bob sends the following quantity to Alice over the public channel:

$$S_L \oplus Q$$

Alice Uses $X^n|_{L_0 \cup \tilde{G}_S}$ to form a secret key S_0 and uses $X^n|_{L_1 \cup \tilde{G}_S}$ to form a secret key S_1 . Both S_0, S_1 are nr -bit each and are secret from Eve. Alice finally sends the following two encrypted strings to Bob over the public channel:

$$K_0 \oplus S_0, \quad K_1 \oplus S_1$$

Bob Knows $X^n|_{L_U \cup \tilde{G}_S}$ and hence knows S_U , thereby recovering K_U from Alice's public message.

Lemma 1. A rate of $\min\{\frac{1}{3}\epsilon_2(1 - \epsilon_1), \epsilon_1\}$ is achievable in the setup of Fig. 2.

A proof of this lemma is deferred to the Appendix. Below, we give a sketch of this proof.

- (2) is satisfied for the following reason. Since Bob knows both $X^n|_{L_U}$ and $X^n|_{\tilde{G}_S}$, Bob knows $X^n|_{L_U \cup \tilde{G}_S}$. Hence, Bob knows S_U and can recover K_U from $K_U \oplus S_U$.
- (3) is satisfied because Bob knows nothing about $X^n|_{L_{\bar{U}}}$. Since $S_{\bar{U}}$ is generated from $X^n|_{L_{\bar{U}} \cup \tilde{G}_S}$ and has the same number of bits as $X^n|_{L_{\bar{U}}}$, Bob will learn practically no information about $S_{\bar{U}}$ and, hence, about $K_{\bar{U}}$.
- Alice can learn about U only from Bob's public messages. In the scheme, Alice learns L_0, L_1 from Bob's public messages. Since L_0, L_1 are of the same size and since the channel acts independently on each input bit, Alice learns no information about U . Hence, (4).
- Finally, Eve cannot learn U since the identity of L_0, L_1 remains hidden from her by the secret key S_L . Eve only

learns $L_0 \cup L_1$ and nothing more. Conditioned on knowing U , Eve still does not learn (K_0, K_1) since these are encrypted using secret keys S_0, S_1 . Hence, (5) is satisfied.

V. ACKNOWLEDGEMENTS

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APPENDIX A

PROOF OF LEMMA 1

In order to prove Lemma 1, we will use a sequence $\{P_n\}_{n \in \mathbb{N}}$ of Protocol 1 and show that for $r < \min\{\frac{1}{3}\epsilon_2(1 - \epsilon_1), \epsilon_1\}$, (2) - (5) hold for $\{P_n\}_{n \in \mathbb{N}}$.

We note that for P_n , the transcript of the public channel is

$$\mathbf{F} = (\tilde{G}, \tilde{B}, S_L \oplus Q, K_0 \oplus S_0, K_1 \oplus S_1). \quad (7)$$

Let J be the indicator random variable for the event that Bob declares an error and quits. Using Chernoff bound, we see that $P[J = 1] \rightarrow 0$ as $n \rightarrow \infty$.

- 1) (2) - (4) hold (see [15, Appendix A] for the proofs).
- 2) In order to show (5) holds, it will suffice to show that $I(K_0, K_1, U; V_E | J = 0) \rightarrow 0$ as $n \rightarrow \infty$. All terms and assertions below are conditioned on the event $J = 0$, but this is being suppressed for ease of writing.

$$\begin{aligned} I(K_0, K_1, U; V_E) &= I(K_U, K_{\bar{U}}, U; V_E) \\ &= I(U; V_E) + I(K_{\bar{U}}; V_E | U) + I(K_U; V_E | U, K_{\bar{U}}) \\ &= I(U; V_E) + I(K_{\bar{U}}; U, V_E) + I(K_U; U, K_{\bar{U}}, V_E) \end{aligned}$$

We will look at each of the above three terms separately.

$$\begin{aligned} I(U; V_E) &= I(U; Z^n, \mathbf{F}) \\ &= I(U; Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_0 \oplus S_0, K_1 \oplus S_1) \\ &\leq I(U; Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_0, S_0, K_1, S_1) \\ &= I(U; Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, S_0, S_1) \\ &[\text{since } K_0, K_1 \text{ are independent of all the variables above}] \\ &= I(U; S_L \oplus Q, S_0, S_1 | Z^n, \tilde{G}, \tilde{B}) \\ &= H(S_L \oplus Q, S_0, S_1 | Z^n, \tilde{G}, \tilde{B}) \\ &\quad - H(S_L \oplus Q, S_0, S_1 | U, Z^n, \tilde{G}, \tilde{B}) \\ &\leq |S_L| + |S_0| + |S_1| - H(S_L, S_0, S_1 | U, Q, Z^n, \tilde{G}, \tilde{B}) \\ &= |S_L| + |S_U| + |S_{\bar{U}}| - H(S_L, S_C, S_{\bar{C}} | U, G, B, Z^n, \tilde{G}, \tilde{B}) \\ &[\text{since } (G, B) \text{ is a function of } (U, Q, \tilde{G}, \tilde{B}) \text{ and } Q \text{ is} \\ &\quad \text{a function of } (U, G, B)] \\ &= |S_L| + |S_U| + |S_{\bar{U}}| \\ &\quad - H(S_L, S_U, S_{\bar{U}} | Z^n|_G, Z^n|_{\tilde{G}_S}, Z^n|_{\tilde{G}_L}) \\ &[\text{since } S_L, S_U, S_{\bar{U}} - Z^n|_G, Z^n|_{\tilde{G}_S}, Z^n|_{\tilde{G}_L} - \\ &\quad U, G, B, Z^n, \tilde{G}, \tilde{B} \text{ is a Markov chain}] \\ &= |S_L| + |S_U| + |S_{\bar{U}}| - H(S_L | Z^n|_G, Z^n|_{\tilde{G}_S}, Z^n|_{\tilde{G}_L}) \\ &\quad - H(S_U, S_{\bar{U}} | S_L, Z^n|_G, Z^n|_{\tilde{G}_S}, Z^n|_{\tilde{G}_L}) \\ &= (|S_L| - H(S_L | Z^n|_{\tilde{G}_L})) \\ &\quad + (|S_U| + |S_{\bar{U}}| - H(S_U, S_{\bar{U}} | Z^n|_G, Z^n|_{\tilde{G}_S})) \\ &[\text{since } S_L - Z^n|_{\tilde{G}_L} - Z^n|_G, Z^n|_{\tilde{G}_S} \text{ and} \\ &\quad S_U, S_{\bar{U}} - Z^n|_G, Z^n|_{\tilde{G}_S} - S_L, Z^n|_{\tilde{G}_L} \text{ are Markov Chains}] \end{aligned}$$

The first term above is small since S_L is a secret key against Eve. Lemma 2 of Appendix B implies that the second term is also small.

$$\begin{aligned} I(K_{\bar{U}}; U, V_E) &= I(K_{\bar{U}}; U, Z^n, \mathbf{F}) \end{aligned}$$

$$\begin{aligned}
&= I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_0 \oplus S_0, K_1 \oplus S_1) \\
&= I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_U \oplus S_U, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&\leq I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_U, S_U, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&= I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, S_C, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&\text{[since } K_U \text{ is independent of all other variables above]} \\
&\leq I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L, Q, S_U, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&= I(K_{\bar{U}}; U, Z^n, \tilde{G}, \tilde{B}, S_L, G, B, S_U, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&\text{[since } (G, B) \text{ is a function of } (U, Q, \tilde{G}, \tilde{B}) \text{ and } Q \text{ is} \\
&\text{a function of } (U, G, B)] \\
&= I(K_{\bar{U}}; S_U, S_L, Z^n|_G, Z^n|_{\tilde{G}}, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&\text{[since } K_{\bar{U}} - S_U, S_L, Z^n|_G, Z^n|_{\tilde{G}}, K_{\bar{U}} \oplus S_{\bar{U}} - \\
&\quad U, Z^n, G, B, \tilde{G}, \tilde{B} \text{ is a Markov chain]} \\
&= I(K_{\bar{U}}; S_U, S_L, Z^n|_G, Z^n|_{\tilde{G}_L}, Z^n|_{\tilde{G}_S}, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&= I(K_{\bar{U}}; S_U, Z^n|_G, Z^n|_{\tilde{G}_S}, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&\text{[since } K_{\bar{U}} - K_{\bar{U}} \oplus S_{\bar{U}}, S_U, Z^n|_G, Z^n|_{\tilde{G}_S} - S_L, Z^n|_{\tilde{G}_L} \\
&\text{is a Markov chain]} \\
&= I(K_{\bar{U}}; S_U, K_{\bar{U}} \oplus S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&= H(S_U, K_{\bar{U}} \oplus S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\quad - H(S_U, S_{\bar{U}} \mid K_{\bar{U}}, Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&= H(S_U, K_{\bar{U}} \oplus S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\quad - H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\leq |S_U| + |S_{\bar{U}}| - H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\text{The above term is small as a consequence of Lemma 2.} \\
&I(K_U; C, K_{\bar{U}}, V_E) \\
&= I(K_U; U, K_{\bar{U}}, Z^n, \mathbf{F}) \\
&= I(K_U; U, K_{\bar{U}}, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_0 \oplus S_0, K_1 \oplus S_1) \\
&= I(K_U; U, K_{\bar{U}}, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_U \oplus S_U, K_{\bar{U}} \oplus S_{\bar{U}}) \\
&= I(K_U; U, K_{\bar{U}}, S_{\bar{U}}, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_U \oplus S_U) \\
&= I(K_U; U, S_{\bar{U}}, Z^n, \tilde{G}, \tilde{B}, S_L \oplus Q, K_U \oplus S_U) \\
&\text{[since } K_{\bar{U}} \text{ is independent of all other variables above]} \\
&\leq I(K_U; U, S_{\bar{U}}, Z^n, \tilde{G}, \tilde{B}, S_L, Q, K_U \oplus S_U) \\
&= I(K_U; U, S_{\bar{U}}, S_L, Z^n, \tilde{G}, \tilde{B}, G, B, K_U \oplus S_U) \\
&\text{[since } (G, B) \text{ is a function of } (U, Q, \tilde{G}, \tilde{B}) \text{ and } Q \text{ is} \\
&\text{a function of } (U, G, B)] \\
&= I(K_U; S_{\bar{U}}, S_L, Z^n|_G, Z^n|_{\tilde{G}}, K_U \oplus S_U) \\
&\text{[since } K_U - S_{\bar{U}}, S_L, Z^n|_G, Z^n|_{\tilde{G}}, K_U \oplus S_U - \\
&\quad U, Z^n, G, B, \tilde{G}, \tilde{B} \text{ is a Markov chain]} \\
&= I(K_U; S_{\bar{U}}, S_L, Z^n|_G, Z^n|_{\tilde{G}_L}, Z^n|_{\tilde{G}_S}, K_U \oplus S_U) \\
&= I(K_U; S_{\bar{U}}, Z^n|_G, Z^n|_{\tilde{G}_S}, K_U \oplus S_U) \\
&\text{[since } K_U - K_U \oplus S_U, S_{\bar{U}}, Z^n|_G, Z^n|_{\tilde{G}_S} - S_L, Z^n|_{\tilde{G}_L} \\
&\text{is a Markov chain]} \\
&= I(K_U; S_{\bar{U}}, K_U \oplus S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S})
\end{aligned}$$

$$\begin{aligned}
&= H(S_{\bar{U}}, K_U \oplus S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\quad - H(S_{\bar{U}}, S_U \mid K_U, Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&= H(S_{\bar{U}}, K_U \oplus S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\quad - H(S_{\bar{U}}, S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\leq |S_{\bar{U}}| + |S_U| - H(S_{\bar{U}}, S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S})
\end{aligned}$$

The above term is small as a consequence of Lemma 2.

APPENDIX B A USEFUL LEMMA

Lemma 2. *Let Alice transmit a sequence X^n of i.i.d. $\text{Ber}(\frac{1}{2})$ bits and let Bob define the sets G, B, \tilde{G}_S as in Protocol 1. There exists a sequence of codes $\{\mathcal{M}_n^U, \mathcal{M}_n^{\bar{U}}\}_{n \in \mathbb{N}}$, where $\mathcal{M}_n^U : (X^n|_{\tilde{G}_S}, X^n|_G) \mapsto S_U$ and $\mathcal{M}_n^{\bar{U}} : (X^n|_{\tilde{G}_S}, X^n|_B) \mapsto S_{\bar{U}}$ such that $|S_U| = |S_{\bar{U}}| = nr$ and, for $n \rightarrow \infty$,*

- 1) $H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) - 2nr \rightarrow 0$,
- 2) $H(S_{\bar{U}} \mid X^n|_{\tilde{G}_S}) - nr \rightarrow 0$.

Proof: We prove this lemma using a random coding argument. The codes (maps) \mathcal{M}_n^U and $\mathcal{M}_n^{\bar{U}}$ are chosen independently and uniformly at random from among all possible maps. We treat the entropies $H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S})$ and $H(S_{\bar{U}} \mid X^n|_{\tilde{G}_S})$ as random variables (which depend on the random code). It then suffices to show that with high probability they approach $2nr$ and nr , respectively.

We will make use of Lemma 3, which is stated after this proof. Consider,

$$\begin{aligned}
&H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&= H(S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) + H(S_{\bar{U}} \mid S_U, Z^n|_G, Z^n|_{\tilde{G}_S}) \\
&\geq H(S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) + H(S_{\bar{U}} \mid X^n|_G, X^n|_{\tilde{G}_S}).
\end{aligned}$$

For the second term, we may directly invoke Lemma 3 using the fact that $|B| > |S_{\bar{U}}| = nr$ to conclude that with high probability the second term approaches nr . For the first term, let Υ be the typical event that the fraction of erasures in $(Z^n|_G, Z^n|_{\tilde{G}_S})$ is at least $\epsilon_2(1 - \frac{\delta}{2})$.

$$H(S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}) \geq H(S_U \mid Z^n|_G, Z^n|_{\tilde{G}_S}, \Upsilon)P(\Upsilon).$$

Using the fact that $P(\Upsilon) \rightarrow 1$ and invoking Lemma 3 we may conclude that the first term also approaches nr with high probability.

We may directly invoke Lemma 3 as we did above for the second term to conclude that $H(S_{\bar{U}}|X^n|_{\tilde{G}_S}) \rightarrow nr$ with high probability. Hence, by union bound we can conclude that with high probability $H(S_U, S_{\bar{U}} \mid Z^n|_G, Z^n|_{\tilde{G}_S})$ and $H(S_{\bar{U}} \mid X^n|_{\tilde{G}_S})$ approach $2nr$ and nr , respectively. ■

Lemma 3 ([15]). *Let $\alpha, \beta, \delta > 0$ be such that $\alpha + \beta + \delta < 1$. Let X be a vector chosen uniformly at random from $\{0, 1\}^n$. Let $f : \{0, 1\}^n \rightarrow \{0, 1\}^{n\alpha}$ be a map chosen uniformly at random from all possible maps. Then with high probability over the choice of f , the map satisfies the property that for every $I \subset \{1, 2, \dots, n\}$ with $|I| \leq n\beta$, and for every $y \in \{0, 1\}^{|I|}$, $H(f(X)|X|_I = y) \geq n\alpha - 2^{-\delta n}$.*