

# Algebraic Properties of Space-Time Block Codes in Inter-Symbol Interference Multiple Access Channels

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## Abstract

*In this paper we study the multiple-access channel where users employ space-time block codes (STBC). The problem is formulated in the context of an inter-symbol interference (ISI) multiple access channel which occurs for transmission over frequency-selective channels. The algebraic structure of the STBC is utilized to design joint interference suppression, equalization, and decoding schemes. Each of the  $K$  users transmits using  $M_t = 2$  transmit antennas and a time-reversed space-time block code suitable for frequency-selective channels. We first show that a diversity order of  $2M_r(\nu + 1)$  is achievable at full transmission rate for each user, when we have  $M_r$  receive antennas, channel memory of  $\nu$  and an optimal multiuser maximum-likelihood (ML) decoder is used. Due to the decoding complexity of the ML detector we study the algebraic structure of linear multiuser detectors which utilize the properties of the STBC. We do this both in the transform ( $D$ -domain) formulation and when we impose finite block length constraints (matrix formulation). The receiver is designed to utilize the algebraic structure of the codes in order to preserve the block quaternionic structure of the equivalent channel for each user. We also explore some algebraic properties of  $D$ -domain quaternionic matrices and of quaternionic block circulant matrices that arise in this study.*

## 1 Introduction

In wireless communication networks, frequency spectrum is a scarce resource that should be efficiently utilized. Since their invention, space-time block codes (STBC) [5] have been shown to have the potential to significantly increase the rates and spectral efficiency of wireless transmissions. Given the limited spectral resources, in this paper we consider multiple co-channel users each equipped with two transmit antennas sharing a frequency-selective channel. The goal is to design space-time transmitter and receiver techniques that allow for efficient detection of the co-channel users while realizing rate and diversity gains without bandwidth expansion.

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In this paper, we consider time-reversed space-time transmitter and receiver techniques in multiuser environments. Multiuser detection has been a rich area of research with many results related to Code Division Multiple Access (CDMA) systems (see for example [14] for more information on this topic). Our interest in this paper is on multiple-antenna transmitters and receivers which employ STBC at the transmitter. The system configuration we are interested in is illustrated in Figure 1 in the two-user scenario. The users are each equipped with multiple transmit antennas, and are transmitting simultaneously over the common multiple access channel to the receiver which has multiple receive antennas. The need for efficient utilization of available transmission bandwidth motivates such a system configuration. The question is whether we can utilize the space-time coded structure of transmissions to ease the multiuser detection problem at the receiver. This leads to the problem of the receiver being able to efficiently perform multipacket reception. From the perspective of network operator, one would ideally like to pack as many users as possible without suffering in performance. Therefore the goal is to devise a transmission and reception strategy for the multiuser system with complexity not much greater than a single-user system but with minimal performance loss. This is a challenging problem, especially in the presence of an inter-symbol interference multiple access channel. A subtext to this question is to quantify the gains in performance one can obtain, by placing multiple antennas at both ends of an ISI multiple access channel.

The interference cancellation technique presented in [11] for flat-fading channels can be directly extended to frequency-selective channels by combining it with either orthogonal frequency division multiplexing (OFDM) or with a single-carrier frequency-domain equalizer (SC-FDE) [2]. There are three main reasons for considering the time-domain single-carrier technique. The first and perhaps the most important reason is that there is a simple technique to ensure both spatial and multipath diversity gains for two transmit antennas without rate loss.<sup>1</sup> Second, OFDM suffers from the problems of high peak-to-average power ratio and increased sensitivity to frequency synchronization errors. Single-carrier techniques do not suffer from this problem. The last reason is that the performance of the time-domain techniques is better than OFDM and SC-FDE for uncoded systems [4] and the three schemes have comparable performance for coded systems. All these reasons motivate the investigation of time-domain techniques. Another significant motivation for this study is that interesting algebraic properties arise making them worthwhile to examine from a theoretical point of view.

Previous related work includes space-time interference cancellation techniques for flat-fading channels in [11], joint frequency-domain zero-forcing interference cancellation and equalization for frequency-selective channels in [2], and extensions to the case of more than two users and more than two transmit antennas in [1].

The main contribution of this paper is identification of some key algebraic properties that

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<sup>1</sup>Except for the rate loss associated with the guard sequence which is common to all block transmission schemes over ISI channels.

allows both a simple derivation of the receiver techniques and exposes some properties of the overall system. The key algebraic property we identify is a multiplicative group property of both D-domain quaternionic matrices as well as block circulant matrices. The consequences of these properties are investigated and utilized throughout the paper. We first start with the multiuser ML decoder for this problem and prove that a diversity order of  $2M_r(\nu + 1)$  is achievable for each of the  $K$  users transmitting at full rate using  $M_t = 2$  transmit antennas over ISI channels of memory  $\nu$  when we use  $M_r$  receive antennas. Due to the complexity of the multiuser ML detector we study the design of linear multiuser receivers in frequency-selective channels which utilize the special STBC structure induced by the transmitters. We demonstrate this technique both in the context of transform domain designs (suitable for serial transmissions) and in matrix formulations (suitable for finite block transmissions) without having issues with edge effects. These finite block length matrix formulations also lead to receiver structures that can be implemented efficiently through finite-impulse response (FIR) processing. Though it is not the focus of this paper, we observe that all these methods can easily be combined with iterative soft-decision receivers to further improve performance.

This paper is organized as follows. In Section 2 we introduce the transmission technique used and set up the notation for both the D-domain discussion and the finite block length scenarios. In Section 3, we develop the joint ML decoder and present the diversity order result for the optimal decoder. In Section 4, we develop the linear multiuser detector in D-domain framework which illustrates the algebraic properties used. In Section 5, we present the finite block length implementation of the joint space-time interference suppression and equalization scheme which exposes some algebraic properties of quaternionic block circulant matrices. The paper is concluded in Section 6 with some of the detailed proofs relegated to the appendices.

## 2 Input-Output Model

In this section, we present the input-output model for both single-user and multi-user scenarios under both serial (D-domain) and finite-block (matrix) transmission conditions.

We transmit information by encoding over two transmission blocks each of length  $N$  (see Figure 2) over which the channel is assumed to be quasi-static. In addition,  $\nu$  zero symbols are inserted as guard between data blocks to eliminate inter-block interference.

## 2.1 D-domain Formulation

In the D-transform notation<sup>2</sup>, the received sequences  $y_1(D), y_2(D)$  for the first and second sub-blocks are given by

$$\begin{aligned} y_1(D) &= h_{1,1}(D)c_{1,1}(D) + h_{1,2}(D)c_{2,1}(D) + \eta_1(D) \\ y_2(D) &= h_{1,1}(D)c_{1,2}(D) + h_{1,2}(D)c_{2,2}(D) + \eta_2(D), \end{aligned} \quad (1)$$

where  $c_{i,l}(D)$  denote the transmitted sequences from the  $i$ th transmit antenna for the  $l$ th sub-block, where  $i = 1, 2$  and  $l = 1, 2$ ,  $h_{i,j}(D)$  is the channel from the  $j$ th transmit antenna to the  $i$ th receive antenna, and  $\eta_l(D)$  (for  $l = 1, 2$ ) are the noise sequences. The channels are assumed to be FIR filters with memory  $\nu$ . Throughout this paper we assume that the noise processes<sup>3</sup>  $\eta_i(D)$ ,  $i = 1, 2$  are zero-mean Gaussian with a unit-variance white power spectrum, *i.e.*,<sup>4</sup>  $\mathbb{E}[\eta_i(D)\bar{\eta}_i(\bar{D}^{-1})] = 1$ , for  $i = 1, 2$ . Also, the data sequences are assumed to be white in deriving the MMSE suppression scheme in Section 4.3.2. Finally, the channel responses  $h_{i,j}(D)$ ,  $i, j = 1, 2$  are assumed to be independent complex Gaussian with unit energy (across all taps) with independent and identically distributed coefficient for each channel tap (*i.e.*, a Rayleigh fading wide-sense stationary uncorrelated scattering channel model [8] with  $\mathbb{E}[h_{i,j}(D)\bar{h}_{i,j}(\bar{D}^{-1})] = 1$ , for  $i, j = 1, 2$ ). The finite-block vector model is also developed later in this section. For two information sequences  $\{\mathbf{c}_1[n]\}, \{\mathbf{c}_2[n]\}$ , we transmit the sequences and the time-reversed conjugated versions ( $\{\bar{\mathbf{c}}_1[-n]\}, \{\bar{\mathbf{c}}_2[-n]\}$ ) over the sub-blocks as shown below (this is the so-called time-reversal (TR)-STBC technique introduced in [10]). Over the transmission block, we can write the D-transform of the received sequence as

$$\begin{aligned} \mathbf{y}(D) &= \begin{bmatrix} y_1(D) & y_2(D) \end{bmatrix} \\ &= \begin{bmatrix} h_{1,1}(D) & h_{1,2}(D) \end{bmatrix} \begin{bmatrix} c_1(D) & c_2(D) \\ -\bar{c}_2(\bar{D}^{-1}) & \bar{c}_1(\bar{D}^{-1}) \end{bmatrix} + \begin{bmatrix} \eta_1(D) \\ \eta_2(D) \end{bmatrix}, \end{aligned} \quad (2)$$

where  $\bar{c}[-n] \iff \bar{c}(\bar{D}^{-1})$  indicates conjugated time-reversed sequences. Defining,  $\mathbf{r}_1(D) = [y_1(D), -\bar{y}_2(\bar{D}^{-1})]^T$ , we can write

$$\begin{aligned} \mathbf{r}_1(D) &= \begin{bmatrix} h_{1,1}(D) & h_{1,2}(D) \\ -\bar{h}_{1,2}(\bar{D}^{-1}) & \bar{h}_{1,1}(\bar{D}^{-1}) \end{bmatrix} \begin{bmatrix} c_1(D) \\ c_2(D) \end{bmatrix} + \begin{bmatrix} \eta_1(D) \\ -\bar{\eta}_2(\bar{D}^{-1}) \end{bmatrix} \\ &\stackrel{def}{=} \mathbf{H}_1(D)\mathbf{c}(D) + \boldsymbol{\eta}_1(D). \end{aligned} \quad (3)$$

This model is now easily extended to the 2 receive antenna case by denoting  $\mathbf{r}(D) = [\mathbf{r}_1(D), \mathbf{r}_2(D)]^T$  where we obtain

$$\mathbf{r}(D) = \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \mathbf{c}(D) + \begin{bmatrix} \boldsymbol{\eta}_1(D) \\ \boldsymbol{\eta}_2(D) \end{bmatrix}, \quad (4)$$

<sup>2</sup>The D-transform is identical to the well-known Z-transform [12] with  $D = z^{-1}$ .

<sup>3</sup>For a sequence  $\{c_0, c_1, \dots\}$ ,  $c(D) = c_0 + c_1D + \dots$  and  $\bar{c}(\bar{D}^{-1}) = \bar{c}_0 + \bar{c}_1\bar{D}^{-1} + \dots$

<sup>4</sup>In this paper, for a complex matrix (or vector)  $\mathbf{A}$  we denote by  $\mathbf{A}^T$  its transpose, by  $\bar{\mathbf{A}}$  its Hermitian transpose, and by  $\mathbf{A}^*$  its complex conjugate.

where  $\mathbf{H}_2(D)$  is the channel transfer matrix to the second receive antenna.

Now, for the two-user case, denoting the corresponding channel transfer matrices for second user by  $\mathbf{G}_1(D)$  and  $\mathbf{G}_2(D)$ , we obtain

$$\mathbf{r}(D) = \begin{bmatrix} \mathbf{H}_1(D) & \mathbf{G}_1(D) \\ \mathbf{H}_2(D) & \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \mathbf{c}(D) \\ \mathbf{s}(D) \end{bmatrix} + \begin{bmatrix} \eta_1(D) \\ \eta_2(D) \end{bmatrix}. \quad (5)$$

Finally, in the case of  $M_r$  receive antennas and  $K$  users each using  $M_t = 2$  transmit antennas, (5) can be generalized as follows

$$\begin{bmatrix} \mathbf{r}_1(D) \\ \mathbf{r}_2(D) \\ \vdots \\ \mathbf{r}_{M_r}(D) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1^{(1)}(D) & \dots & \mathbf{H}_1^{(K)}(D) \\ \mathbf{H}_2^{(1)}(D) & \dots & \mathbf{H}_2^{(K)}(D) \\ \vdots & \vdots & \vdots \\ \mathbf{H}_{M_r}^{(1)}(D) & \dots & \mathbf{H}_{M_r}^{(K)}(D) \end{bmatrix} \begin{bmatrix} \mathbf{c}^{(1)}(D) \\ \mathbf{c}^{(2)}(D) \\ \vdots \\ \mathbf{c}^{(K)}(D) \end{bmatrix} + \begin{bmatrix} \eta_1(D) \\ \eta_2(D) \\ \vdots \\ \eta_{M_r}(D) \end{bmatrix}, \quad (6)$$

where  $\mathbf{H}_m^{(k)}(D)$  is the channel from the  $k$ th user to the  $m$ th receive antenna, and  $\mathbf{c}^{(k)}(D)$  is the data sequence of the  $k$ th user. Many of the receiver structures of this paper are illustrated using the two-user two-receive-antenna case. For simplicity, we consider the case of equal-power users (*i.e.* 0 dB SIR), and the extension to arbitrary SIR is straightforward.

## 2.2 Finite Block Length Matrix Formulation

Our starting point in developing the FIR form for the single-user scenario is the representation of the input-output relationship in (2) in the following matrix form

$$\mathbf{y}_1 = \mathbf{H}_{1,1} \mathbf{I}_{zs} \mathbf{c}_{1,1} + \mathbf{H}_{1,2} \mathbf{I}_{zs} \mathbf{c}_{2,1} + \eta_1, \quad (7)$$

where  $\mathbf{H}_{1,1}$  and  $\mathbf{H}_{1,2}$  are square  $(N + \nu)$ -dimensional lower-triangular Toeplitz matrices whose first columns are equal to the  $(\nu + 1)$  impulse response coefficients of  $h_{1,1}(D)$  and  $h_{1,2}(D)$  appended by  $(N - 1)$  zeros, respectively. The output and noise vectors  $\mathbf{y}_{1,1}$  and  $\eta_{1,1}$  are  $(N + \nu)$ -dimensional while the data vectors  $\mathbf{c}_{1,1}$  and  $\mathbf{c}_{2,1}$  are  $N$ -dimensional. This matrix model assumes the insertion of  $\nu$  zeros at the end of each data vector to eliminate inter-block interference. This **zero-stuffing** operation is represented in (7) by the matrix  $\mathbf{I}_{zs} \stackrel{def}{=} \begin{bmatrix} \mathbf{I}_N \\ \mathbf{0}_{\nu \times N} \end{bmatrix}$ . The output of the second sub-block is given by

$$\mathbf{y}_2 = \mathbf{H}_{1,1} \mathbf{I}_{zs} \mathbf{J}_N \mathbf{c}_{1,2}^* - \mathbf{H}_{1,2} \mathbf{I}_{zs} \mathbf{J}_N \mathbf{c}_{1,1}^* + \eta_2, \quad (8)$$

where  $\mathbf{J}_N$  is the  $N$ -dimensional reversal matrix that consists of ones on the anti-diagonal and zeros everywhere else.

Conjugating and reversing  $\mathbf{y}_2$  and combining it with  $\mathbf{y}_1$ , we get the following space–time FIR model for (3)

$$\begin{aligned} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{J}_{N+\nu}\mathbf{y}_2^* \end{bmatrix} &= \begin{bmatrix} \mathbf{H}_{1,1}^{zs} & \mathbf{H}_{1,2}^{zs} \\ -\mathbf{J}_{N+\nu}\mathbf{H}_{1,2}^{*zs}\mathbf{J}_N & \mathbf{J}_{N+\nu}\mathbf{H}_{1,1}^{*zs}\mathbf{J}_N \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \mathbf{J}_{N+\nu}\eta_2^* \end{bmatrix} \\ \Rightarrow \mathbf{r}_1 &\stackrel{def}{=} \mathbf{H}_1\mathbf{c} + \eta_1, \end{aligned} \quad (9)$$

where the superscript  $zs$  on a matrix indicates multiplication by the zero–stuffing matrix  $\mathbf{I}_{zs}$ . The overall channel matrix  $\mathbf{H}_1$  is of size  $2(N + \nu) \times 2N$  and the processed output  $\mathbf{r}_1$  is a vector of size  $2(N + \nu)$ . Note that pre– and post–multiplication of the channel matrices  $\mathbf{H}_{1,1}^{*zs}$  and  $\mathbf{H}_{1,2}^{*zs}$  by the reversal matrices  $\mathbf{J}_{N+\nu}$  and  $\mathbf{J}_N$  results in lower–triangular Toeplitz matrices whose first columns are equal to the time–reversed and conjugated coefficients of  $h_{1,1}(D)$  and  $h_{1,2}(D)$ , as desired.

For the multi-user case, it turns out that the output blocks need to be processed in a manner different from (9) (see Section 5.2 for more details). More specifically, for the two-user case, by applying a different linear transformation which performs a partial reversal of the second sub-block, it is shown in [15] that the following finite-block length form is obtained.

$$\begin{aligned} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{P}\mathbf{y}_2^* \end{bmatrix} &= \begin{bmatrix} \mathbf{H}_{1,1}^{(c)} & \mathbf{H}_{1,2}^{(c)} \\ -\bar{\mathbf{H}}_{1,2}^{(c)} & \bar{\mathbf{H}}_{1,1}^{(c)} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{zs}\mathbf{c}_1 \\ \mathbf{I}_{zs}\mathbf{c}_2 \end{bmatrix} + \begin{bmatrix} \eta_1 \\ \mathbf{P}\eta_2^* \end{bmatrix} \\ \Rightarrow \mathbf{r}_1^{(c)} &\stackrel{def}{=} \mathbf{H}_1^{(c)}\mathbf{c} + \eta_1^{(c)}, \end{aligned} \quad (10)$$

where the matrices  $\mathbf{H}_{1,k}^{(c)}$ ,  $k = 1, 2$  represent the  $(N + \nu)$ -dimensional square circulant matrices derived<sup>5</sup> from  $\mathbf{H}_{1,k}$ ,  $k = 1, 2$ .

The matrix  $\mathbf{P}$  is a partial permutation matrix<sup>6</sup> [15].

Finally, for the case of  $M_r$  receive antennas and  $K$  users each using  $M_t = 2$  transmit antennas, (10) can be generalized as follows

$$\mathbf{r} = \begin{bmatrix} \mathbf{r}_1^{(c)} \\ \mathbf{r}_2^{(c)} \\ \vdots \\ \mathbf{r}_{M_r}^{(c)} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{H}_1^{(1)} & \dots & \mathbf{H}_1^{(K)} \\ \mathbf{H}_2^{(1)} & \dots & \mathbf{H}_2^{(K)} \\ \vdots & \vdots & \vdots \\ \mathbf{H}_{M_r}^{(1)} & \dots & \mathbf{H}_{M_r}^{(K)} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mathbf{c}^{(1)} \\ \mathbf{c}^{(2)} \\ \vdots \\ \mathbf{c}^{(K)} \end{bmatrix}}_{\mathbf{c}} + \underbrace{\begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{M_r} \end{bmatrix}}_{\boldsymbol{\eta}} \quad (11)$$

where  $\mathbf{H}_m^{(k)}$ ,  $k = 1, \dots, K$ ,  $m = 1, \dots, M_r$  is the channel matrix from the  $k$ th user to the  $m$ th receive antenna. Therefore, the matrix  $\mathbf{H}_m^{(k)}$  has the same form as the matrix  $\mathbf{H}_1^{(c)}$  in (10), *i.e.* a block circulant structure which will be utilized extensively in this paper. Finally,  $\mathbf{c}^{(k)}$  is the data vector of the  $k$ th user.

<sup>5</sup>The matrices  $\mathbf{H}_{1,k}^{(c)}$  are circulant matrices which have the same first column as the lower triangular Toeplitz matrices  $\mathbf{H}_{1,k}$  described in (7). Note that  $\mathbf{H}_{1,k}^{(c)}$  are  $(N + \nu)$ -dimensional square matrices, unlike the  $N$ -dimensional square circulant matrices more familiarly encountered in Discrete Multitone Transmissions (DMT) [6].

<sup>6</sup>For  $\mathbf{a}^T = \begin{bmatrix} \mathbf{a}(0) & \dots & \mathbf{a}(N + \nu - 1) \end{bmatrix}$ ,  $\mathbf{P}\mathbf{a} = \begin{bmatrix} \mathbf{a}(N - 1), & \dots, & \mathbf{a}(0), & \mathbf{a}(N + \nu - 1), & \dots, & \mathbf{a}(N) \end{bmatrix}^T$ .

### 3 Diversity Order of ML Decoder

Given the multiuser input-output models in (6) and (11), we can develop the optimal joint multiuser detector based on maximum-likelihood (ML) decoding [14]. We will illustrate this using the matrix model of (11). The ML decoding metric is

$$\{\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \dots, \mathbf{c}^{(K)}\} = \operatorname{argmin}_{\mathbf{c}} \|\mathbf{r} - \mathbf{H}\mathbf{c}\|^2 \quad (12)$$

which is computed using a joint trellis implementing the Viterbi algorithm [14].

The notion of diversity order for space-time codes has been defined in [13] as follows.

**Definition 3.1** *A coding scheme which has an average error probability  $\bar{P}_e(SNR)$  as a function of SNR that behaves as*

$$\lim_{SNR \rightarrow \infty} \frac{\log(\bar{P}_e(SNR))}{\log(SNR)} = -d \quad (13)$$

*is said to have a diversity order of  $d$ .*

In words, a scheme with diversity order  $d$  has an error probability at high SNR behaving as  $\bar{P}_e(SNR) \approx SNR^{-d}$ . The notion of full transmission rate (as defined in [13]) implies that if we use a constellation size of  $2^b$  for transmission, the space time code sends  $b$  bits/sec/Hz information symbols. For example the STBC defined by Alamouti [5] has full transmission rate since 2 information symbols are sent over two time units.

For the multiple-access channel defined in (11), using the ML decoding metric given in (12), we can prove the following result on the diversity order of TR-STBC transmissions.

**Theorem 3.2** *A multiple-access system with  $M_r$  receive antennas and  $K$  synchronous users each transmitting TR-STBC signals using  $M_t = 2$  antennas over ISI channels with memory  $\nu$  achieves a diversity order of  $2M_r(\nu + 1)$  at full transmission rate for each user.*

The proof of Theorem 3.2 is given in Appendix A. This result implies that if optimum decoding is used then the performance observed by any individual user is equivalent to the system where only that user is transmitting. This is quite satisfying since we know (see [10],[15]) that for a single-user system with  $M_r = 2$ , the TR-STBC achieves the maximal order of diversity  $2M_r(\nu + 1)$ .

In order to achieve the diversity order predicted in Theorem 3.2, we would need to do joint multiuser ML decoding of the  $K$ -user multiple-access ISI channel. This is computationally expensive with the decoding complexity being exponential in the channel length  $\nu$ , the number of users  $K$ , and the spectral efficiency of the signal constellation. This motivates the suboptimal reduced-complexity multiuser linear detector structures described in Sections 4 and 5. These receivers use the algebraic structure of the space-time block code in order to construct efficient detection schemes.

## 4 D-domain Processing

In this section we develop the D-domain processing framework for joint equalization and interference suppression which is suitable for serial transmissions. The finite-length block processing case is developed in Section 5. We start in Section 4.1 by observing some algebraic properties of the model developed in Section 2. Then, we develop linear multiuser detectors, in both the decorrelating case (Section 4.2) and the MMSE case (Section 4.3).

### 4.1 Preliminaries

Define the set  $\mathcal{Q}$  of invertible<sup>7</sup>  $2 \times 2$  D-domain matrices of the form

$$\mathbf{X}(D) \stackrel{def}{=} \begin{bmatrix} X_1(D) & X_2(D) \\ -\bar{X}_2(\bar{D}^{-1}) & \bar{X}_1(\bar{D}^{-1}) \end{bmatrix}. \quad (14)$$

By direct verification we can show the following property.

**Lemma 4.1**  *$\mathcal{Q}$  forms a multiplicative group, i.e., it has the following properties*

$$\text{For } \mathbf{V}_1(D), \mathbf{V}_2(D) \in \mathcal{Q}, \mathbf{V}_1(D)\mathbf{V}_2(D) \in \mathcal{Q}. \quad (15)$$

$$[\mathbf{V}(D)]^{-1} = \frac{1}{\|\mathbf{V}(D)\|^2} \bar{\mathbf{V}}(\bar{D}^{-1}) \in \mathcal{Q}.$$

where

$$\bar{\mathbf{V}}(\bar{D}^{-1}) \stackrel{def}{=} \begin{bmatrix} \bar{V}_1(\bar{D}^{-1}) & -V_2(D) \\ \bar{V}_2(\bar{D}^{-1}) & V_1(D) \end{bmatrix}, \quad (16)$$

and

$$\|\mathbf{V}(D)\|^2 = \mathbf{V}(D)\bar{\mathbf{V}}(\bar{D}^{-1}) = V_1(D)\bar{V}_1(\bar{D}^{-1}) + V_2(D)\bar{V}_2(\bar{D}^{-1}). \quad (17)$$

Note that in (4)  $\mathbf{H}_1(D), \mathbf{H}_2(D) \in \mathcal{Q}$ . Defining the D-domain vectors

$$\begin{bmatrix} \mathbf{h}_1^v(D) & \mathbf{h}_2^v(D) \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{g}_1^v(D) & \mathbf{g}_2^v(D) \end{bmatrix} \stackrel{def}{=} \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix}. \quad (18)$$

The power spectral density,  $\mathbf{M}(D)$ , of the received signal is given by

$$\begin{aligned} \mathbf{M}(D) &\stackrel{def}{=} \mathbb{E}[\mathbf{r}(D)\bar{\mathbf{r}}(\bar{D}^{-1})] \\ &= \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} + \frac{1}{\Gamma} \mathbf{I}_4, \end{aligned} \quad (19)$$

where  $\Gamma$  is the input SNR and we assumed that the input sequences are independent and have a white spectrum.

<sup>7</sup>Note that invertibility is defined in the sense of D-domain matrices (see [9] Section 6.3).



**Definition 4.2 (Special Pair)** Let  $\mathcal{H} = \text{span}(\mathbf{h}_1^v(D), \mathbf{h}_2^v(D))$  and  $\mathcal{G} = \text{span}(\mathbf{g}_1^v(D), \mathbf{g}_2^v(D))$ . We denote a pair of vectors  $\{\mathbf{v}_1^v(D), \mathbf{v}_2^v(D)\} \in \mathcal{H}$  as a special pair if

$$\begin{bmatrix} \mathbf{v}_1^v(D) & \mathbf{v}_2^v(D) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1(D) \\ \mathbf{V}_2(D) \end{bmatrix}, \text{ where } \mathbf{V}_i(D) \in \mathcal{Q}. \quad (20)$$

We define the special pair  $\{\mathbf{x}_1^v(D), \mathbf{x}_2^v(D)\} \in \mathcal{G}$  in a similar manner.

**Definition 4.3** We define the inner product between  $D$ -domain vectors  $\mathbf{h}_1^v(D), \mathbf{h}_2^v(D)$  as

$$\langle \mathbf{h}_1^v(D), \mathbf{h}_2^v(D) \rangle = \bar{\mathbf{h}}_1^v(\bar{D}^{-1}) \mathbf{h}_2^v(D). \quad (21)$$

By direct verification, we can state the following result.

**Lemma 4.4**

$$\langle \mathbf{h}_1^v(D), \mathbf{h}_2^v(D) \rangle = 0 = \langle \mathbf{g}_1^v(D), \mathbf{g}_2^v(D) \rangle. \quad (22)$$

**Lemma 4.5** For any  $\mathbf{F}_i(D) \in \mathcal{Q}$ ,

$$\bar{\mathbf{h}}_1^v(\bar{D}^{-1}) \begin{bmatrix} \mathbf{F}_1(D) \\ \mathbf{F}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{F}}_1(\bar{D}^{-1}) & \bar{\mathbf{F}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{h}_2^v(D) = 0. \quad (23)$$

**Proof:** Using (18),

$$\mathbf{E}(D) \stackrel{\text{def}}{=} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \begin{bmatrix} \mathbf{F}_1(D) \\ \mathbf{F}_2(D) \end{bmatrix} \in \mathcal{Q} \quad (24)$$

and hence  $\mathbf{E}(D) \bar{\mathbf{E}}(\bar{D}^{-1}) = \|\mathbf{E}(D)\|^2 \mathbf{I}_2$ . Since Equation (23) represents the off-diagonal term of  $\mathbf{E}(D) \bar{\mathbf{E}}(\bar{D}^{-1})$ , we obtain the desired result.  $\blacksquare$

Another useful property we will use will be the form for the inverse of  $2 \times 2$  block matrices, which can be verified by direct calculation.

**Lemma 4.6** If  $\mathbf{H}_1(D), \mathbf{H}_2(D), \mathbf{G}_1(D), \mathbf{G}_2(D) \in \mathcal{Q}$ , then

$$\begin{aligned} \mathbf{H}^{-1}(D) &= \begin{bmatrix} \mathbf{H}_1(D) & \mathbf{G}_1(D) \\ \mathbf{H}_2(D) & \mathbf{G}_2(D) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \frac{1}{\|\Delta_{G_2}(D)\|^2} \mathbf{I}_2 & 0 \\ 0 & \frac{1}{\|\Delta_{H_1}(D)\|^2} \mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \|\mathbf{G}_2(D)\|^2 \mathbf{I}_2 & 0 \\ 0 & \|\mathbf{H}_1(D)\|^2 \mathbf{I}_2 \end{bmatrix} \bar{\Delta}(\bar{D}^{-1}), \end{aligned} \quad (25)$$

where

$$\Delta(D) = \begin{bmatrix} \Delta_{G_2}(D) & \Delta_{H_2}(D) \\ \Delta_{G_1}(D) & \Delta_{H_1}(D) \end{bmatrix}, \quad (26)$$

and the  $D$ -domain quantities  $\Delta_{(\cdot)}(D)$  are Schur complements defined as

$$\begin{aligned}\Delta_{G_1}(D) &= \|\mathbf{G}_1(D)\|^2 \mathbf{H}_2(D) - \mathbf{G}_2(D) \bar{\mathbf{G}}_1(\bar{D}^{-1}) \mathbf{H}_1(D), \\ \Delta_{G_2}(D) &= \|\mathbf{G}_2(D)\|^2 \mathbf{H}_1(D) - \mathbf{G}_1(D) \bar{\mathbf{G}}_2(\bar{D}^{-1}) \mathbf{H}_2(D), \\ \Delta_{H_1}(D) &= \|\mathbf{H}_1(D)\|^2 \mathbf{G}_2(D) - \mathbf{H}_2(D) \bar{\mathbf{H}}_1(\bar{D}^{-1}) \mathbf{G}_1(D) \\ \Delta_{H_2}(D) &= \|\mathbf{H}_2(D)\|^2 \mathbf{G}_1(D) - \mathbf{H}_1(D) \bar{\mathbf{H}}_2(\bar{D}^{-1}) \mathbf{G}_2(D).\end{aligned}$$

## 4.2 Zero-Forcing Solution

The zero-forcing solution employs a linear combination of received symbols in order to remove interference between users without any regard to noise enhancement. Consider

$$\mathbf{W}(D) = \begin{bmatrix} \|\mathbf{G}_2(D)\|^2 \mathbf{I}_2 & -\mathbf{G}_1(D) \bar{\mathbf{G}}_2(\bar{D}^{-1}) \\ -\mathbf{H}_2(D) \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \|\mathbf{H}_1(D)\|^2 \mathbf{I}_2 \end{bmatrix}. \quad (27)$$

If we apply the decorrelating matrix filter  $\mathbf{W}(D)$  to the received symbols we obtain

$$\tilde{\mathbf{r}}(D) \stackrel{def}{=} \mathbf{W}(D) \mathbf{r}(D) = \begin{bmatrix} \Delta_{G_2}(D) & \mathbf{0} \\ \mathbf{0} & \Delta_{H_1}(D) \end{bmatrix} \begin{bmatrix} \mathbf{c}(D) \\ \mathbf{s}(D) \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1(D) \\ \tilde{\eta}_2(D) \end{bmatrix}, \quad (28)$$

where due to Lemma 4.6 we can define,

$$\begin{aligned}\Delta_{G_2}(D) &\stackrel{def}{=} \|\mathbf{G}_2(D)\|^2 \mathbf{H}_1(D) - \mathbf{G}_1(D) \bar{\mathbf{G}}_2(\bar{D}^{-1}) \mathbf{H}_2(D) \\ \Delta_{H_1}(D) &\stackrel{def}{=} \|\mathbf{H}_1(D)\|^2 \mathbf{G}_2(D) - \mathbf{H}_2(D) \bar{\mathbf{H}}_1(\bar{D}^{-1}) \mathbf{G}_1(D).\end{aligned} \quad (29)$$

Note that due to Lemma 4.1,  $\Delta_{G_2}(D), \Delta_{H_1}(D) \in \mathcal{Q}$ . Therefore the zero-forcing linear filter  $\mathbf{W}(D)$  decouples the two co-channel users and maintain the structure of the equivalent channel. In particular, each stream of the user (for example  $c_1(D), c_2(D)$  for user 1) can be further decoupled since  $\Delta_{G_2}(D) \bar{\Delta}_{G_2}(\bar{D}^{-1}) = \|\Delta_{G_2}(D)\|^2 \mathbf{I}_2$ . After this decoupling, the users can be equalized (for example through a Viterbi decoder) individually.

Note that the form in (28) produces colored noise, and hence for detection we would need to whiten it. Consider the whitening filter

$$\mathbf{U}(D) = \left[ \|\mathbf{G}_2(D)\| \sqrt{\|\mathbf{G}_1(D)\|^2 + \|\mathbf{G}_2(D)\|^2} \right]^{-1} \mathbf{I}_2, \quad (30)$$

applied to the output of  $\tilde{\mathbf{r}}_1(D) = \Delta_{G_2}(D) \mathbf{c}(D) + \tilde{\eta}_1(D)$ . By examination, this filter whitens the noise  $\tilde{\eta}_1(D)$ . Since  $\mathbf{U}(D) \in \mathcal{Q}$ , this implies that,  $\mathbf{U}(D) \Delta_{G_2}(D) \stackrel{def}{=} \check{\mathbf{H}}(D) \in \mathcal{Q}$  and therefore maintains the structure of the equivalent channel. Hence the whitened output is given by

$$\mathbf{U}(D) \tilde{\mathbf{r}}_1(D) = \check{\mathbf{H}}(D) \mathbf{c}(D) + \mathbf{U}(D) \tilde{\eta}_1(D). \quad (31)$$

The appropriate whitening filter for the stream  $\mathbf{s}(D)$  is similarly  $\left[ \|\mathbf{H}_1(D)\| \sqrt{\|\mathbf{H}_1(D)\|^2 + \|\mathbf{H}_2(D)\|^2} \right]^{-1} \mathbf{I}_2$ . Note that the zero-forcing solution effectively inverts the channel and this is done using the structure of the transmitted space-time code without requiring an explicit channel inversion. In fact, by

using the structure, the decoupling was done using FIR filters. The zero-forcing solution ignores the presence of noise, and therefore is applicable only when the signal-to-noise ratio is high. To overcome this problem we consider a MMSE approach to this problem.

### 4.3 MMSE Interference Suppression

In this section we derive an MMSE receiver that alleviates the noise-enhancement problem of the zero-forcing technique presented in Section 4.2. For this we crucially use the structure of the power spectral density  $\mathbf{M}(D)$  of the received sequence imposed by the transmitter space-time code. An estimate of this power spectral density forms an input to the MMSE solution. Before we present the MMSE technique in Section 4.3.2, we study some properties of  $\mathbf{M}(D)$  in Section 4.3.1.

#### 4.3.1 Properties of $\mathbf{M}(D)$

**Lemma 4.7**

$$\bar{\mathbf{h}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{h}_2^v(D) = 0 = \bar{\mathbf{g}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{g}_2^v(D). \quad (32)$$

More generally, if we have *special pairs*  $\{\mathbf{v}_1^v(D), \mathbf{v}_2^v(D)\} \in \mathcal{H}$  and  $\{\mathbf{x}_1^v(D), \mathbf{x}_2^v(D)\} \in \mathcal{G}$  then,

$$\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{v}_2^v(D) = 0 = \bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{x}_2^v(D). \quad (33)$$

**Proof:**

$$\begin{aligned} \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{v}_2^v(D) &= \bar{\mathbf{v}}_1^v(\bar{D}^{-1}) \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D) \\ &+ \bar{\mathbf{v}}_1^v(\bar{D}^{-1}) \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D) + \frac{1}{\Gamma} \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{v}_2^v(D) \\ &= 0 \end{aligned} \quad (34)$$

where the last equality is due to Lemma 4.5. Similarly, we can prove that  $\bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}(D)\mathbf{x}_2^v(D) = 0$ .

■

The proof of the following theorem is given in Appendix B

**Theorem 4.8** For  $\mathbf{v}_i^v(D) \in \mathcal{H}$  and  $\mathbf{x}_i^v(D) \in \mathcal{G}$ ,

$$\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^k(D)\mathbf{v}_2^v(D) = 0 = \bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}^k(D)\mathbf{x}_2^v(D), \quad \forall k \geq 0. \quad (35)$$

■

From (19) it is clear that  $\mathbf{M}(D)$  is positive definite and therefore is invertible<sup>8</sup>. It follows that  $\mathbf{M}^{-1}(D)$  is polynomial in  $\mathbf{M}(D)$  and therefore the result follows for all negative exponents as well. In particular, we have

$$\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^{-1}(D)\mathbf{v}_2^v(D) = 0 = \bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}^{-1}(D)\mathbf{x}_2^v(D). \quad (36)$$

<sup>8</sup>Note that  $\mathbf{M}(D)$  is nonsingular in the sense of  $\mathbb{D}$ -domain matrices (see [9] Section 6.3).

### 4.3.2 MMSE Interference Suppression

The MMSE interference suppression receiver is found by minimizing the following criterion<sup>9</sup>

$$\mathcal{J}(\alpha(D), \beta(D)) = \mathbb{E}[\|\bar{\alpha}(\bar{D}^{-1})\mathbf{r}(D) - \bar{\beta}(\bar{D}^{-1})\mathbf{c}(D)\|^2], \quad (37)$$

where  $\alpha(D) = [\alpha_1(D), \alpha_2(D), \alpha_3(D), \alpha_4(D)]^T$  and  $\beta(D) = [\beta_1(D), \beta_2(D)]^T$ . In order to equalize  $c_1(D)$ , we set  $\beta_1(D) = 1$ , and define

$$\begin{aligned} \tilde{\alpha}(D) &= [\alpha_1(D), \alpha_2(D), \alpha_3(D), \alpha_4(D) - \beta_2(D)]^T \\ \tilde{\mathbf{r}}(D) &= [r_1(D), r_2(D), r_3(D), r_4(D) - c_2(D)]^T. \end{aligned} \quad (38)$$

Now the optimization problem is

$$\begin{aligned} \min_{\tilde{\alpha}(D)} \{ \mathbb{E}[\|\tilde{\alpha}(\bar{D}^{-1})\tilde{\mathbf{r}}(D) - c_1(D)\|^2] \} = \\ \min_{\tilde{\alpha}(D)} \{ \tilde{\alpha}(\bar{D}^{-1})\mathbb{E}[\tilde{\mathbf{r}}(D)\tilde{\mathbf{r}}(\bar{D}^{-1})]\tilde{\alpha}(D) + 1 - \tilde{\alpha}(\bar{D}^{-1})\mathbb{E}[\tilde{\mathbf{r}}(D)\tilde{c}_1(D)] - \mathbb{E}[c_1(D)\tilde{\mathbf{r}}(\bar{D}^{-1})]\tilde{\alpha}(D) \}. \end{aligned} \quad (39)$$

We can easily verify that

$$\begin{aligned} \mathbb{E}[\tilde{\mathbf{r}}(D)\tilde{\mathbf{r}}(\bar{D}^{-1})] &= \begin{bmatrix} \mathbf{H}(D)\bar{\mathbf{H}}(\bar{D}^{-1}) + \frac{1}{\Gamma}\mathbf{I}_4 & \mathbf{h}_2^v(D) \\ \bar{\mathbf{h}}_2^v(\bar{D}^{-1}) & 1 \end{bmatrix} + \frac{1}{\Gamma} \begin{bmatrix} \mathbf{I}_4 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}. \\ \mathbb{E}[\tilde{\mathbf{r}}(D)\tilde{c}_1(D)] &= \begin{bmatrix} \mathbf{h}_1^v(D) \\ 0 \end{bmatrix}. \end{aligned} \quad (40)$$

Using (40) we can solve (39) in the standard manner to obtain

$$\alpha(D) = \mathbf{M}^{-1}(D)\mathbf{h}_1^v(D), \quad \beta_2(D) = 0. \quad (41)$$

Similarly, in (37) if we set  $\beta_2(D) = 1$ , we would obtain

$$\alpha(D) = \mathbf{M}^{-1}(D)\mathbf{h}_2^v(D), \quad \beta_1(D) = 0, \quad (42)$$

which allows us to equalize  $c_2(D)$  while suppressing the interference. Therefore, in order to decode  $c_1(D)$  we use the scheme prescribed in (41) to obtain

$$\begin{aligned} \bar{\alpha}(\bar{D}^{-1})\mathbf{r}(D) &= \bar{\mathbf{h}}_1^v(\bar{D}^{-1})\mathbf{M}^{-1}(D)\mathbf{h}_1^v(D)c_1(D) \\ &+ \bar{\mathbf{h}}_1^v(\bar{D}^{-1})\mathbf{M}^{-1}(D) \left\{ \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \mathbf{s}(D) + \begin{bmatrix} \eta_1(D) \\ \eta_2(D) \end{bmatrix} \right\}. \end{aligned} \quad (43)$$

We observe that the MMSE detector decouples the streams  $\mathbf{c}_1(D)$  and  $\mathbf{c}_2(D)$  while suppressing the interference from  $\mathbf{s}(D)$ . Note that in the joint equalization and interference suppression

<sup>9</sup>This criterion does joint equalization and interference suppression. However, we will see later that the solution can be split so that the interference suppression and equalization can be separated.

criterion, in order to get to (43) we do not need explicit knowledge of whether or not an interferer is present. All that is required is an estimate of the power spectral density of the received signal sequence.

In order to demonstrate that the interference suppression and equalization can be separated, we can use Lemma 4.6 for writing  $\mathbf{M}^{-1}(D)$ . Using (19) it is clear that  $\mathbf{M}(D) = \begin{bmatrix} \mathbf{P}_1(D) & \mathbf{P}_2(D) \\ \bar{\mathbf{P}}_2(\bar{D}^{-1}) & \mathbf{P}_3(D) \end{bmatrix}$ , where  $\mathbf{P}_i(D) \in \mathcal{Q}$ . Therefore, by multiplying both sides of (43) by  $\mathbf{\Delta}_{P_1}(D)\mathbf{\Delta}_{P_3}(D)$ , we obtain a form which consists of an ‘‘unequalized’’ FIR form for  $c_1(D)$ . This can be used for decoding of  $c_1(D)$ , using any standard technique. A similar argument can be used for  $c_2(D), s_1(D), s_2(D)$ . As the signal-to-noise ratio becomes high, the MMSE receiver reduces to the zero-forcing solution.

## 5 Finite Block Length Case

In practice, it is desirable to implement transmitter and receiver structures using finite block lengths. The development in the D-domain in Section 4 does not clarify whether edge effects in such a scenario would play an important role. The main point of this section is to develop the finite block length analog of Section 4. The D-domain forms, for the most part, are a compact notation for the operations in the finite block length case. Circulant matrices will play the role here of the D-domain polynomials of Section 4. The finite block length processing also allows design of finite-impulse response (FIR) receiver structures which are desirable due to their better numerical properties and suitability for VLSI and programmable DSP implementations as compared to IIR implementations. Note that in this section we will not repeat several of the properties developed in Section 4, each of them has a matrix analog which can be easily derived. We will mention the mapping that allows us the derivation of those properties.

### 5.1 Single-User Scenario

Decoding proceeds by multiplying  $\mathbf{r}_1$  in (9) by the *matrix matched filter*  $\bar{\mathbf{H}}_1$  which is shown in Appendix C to decouple  $\mathbf{c}_1$  and  $\mathbf{c}_2$  while ensuring that the two noise components remain uncorrelated. Hence,  $\mathbf{c}_1$  and  $\mathbf{c}_2$  can be decoded independently without loss of optimality. Moreover, we show in Appendix C that the output of the matrix matched filtering operation is given by

$$\bar{\mathbf{H}}_1 \mathbf{r}_1 = \begin{bmatrix} \mathbf{H}_{eqv} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \mathbf{H}_{eqv} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{z}}_1 \\ \tilde{\mathbf{z}}_2 \end{bmatrix}, \quad (44)$$

where the *equivalent* channel matrix is given by

$$\mathbf{H}_{eqv} = \bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,1}^{zs} + \bar{\mathbf{H}}_{1,2}^{zs} \mathbf{H}_{1,2}^{zs}. \quad (45)$$

Now,  $\mathbf{c}_1$  and  $\mathbf{c}_2$  can be detected using any of several well-known low-complexity detectors such as MMSE block linear or decision-feedback equalizers [3]. It can easily shown that the  $N \times N$

matrix  $\mathbf{H}_{eqv}$  is *Toeplitz*<sup>10</sup> which reduces the complexity of inverting it (to compute the block linear equalizer) or factorizing it using the Levinson or Schur algorithms (to compute the block decision feedback equalizer) by an order of magnitude. As the block length  $N$  becomes infinite, we can invoke the *Toeplitz eigenvalue distribution theorem* [7] to prove that the coefficients of the first column of  $\mathbf{H}_{eqv}$  converge to the coefficients of the correlation sequence  $h_{1,1}(D)\bar{h}_{1,1}(\bar{D}^{-1}) + h_{1,2}(D)\bar{h}_{1,2}(\bar{D}^{-1})$ .

## 5.2 Multi-User Scenario

In the multiuser scenario, which is the focus of this paper, the output processing technique of (9) would need to be modified. This is because the group property used in Lemma 4.1 for D-domain matrices does not hold for the rectangular matrices  $\mathbf{H}_1$  defined in (9). To illustrate this, we assume that we still employ the zero-forcing decorrelating receiver to decouple the two users, followed by a matched filter for each user to decouple the two streams corresponding to its first and second transmit antennas. Since the equivalent channels for each user (after the decorrelating receiver) is not orthogonal, there will be inter-antenna interference which manifests its self as energy in the off-diagonal blocks of the matrix  $\bar{\mathbf{H}}\mathbf{H}$ . In Figure 4, we plot the ratio of this interference energy to the signal energy (i.e. energy in the main diagonal blocks)<sup>11</sup> as a function of the block length  $N$ . As expected, the effects of inter-antenna interference diminish as  $N$  increases and (heuristically) in the limit as  $N$  becomes infinite, the matrix form converges to the polynomial form where we have perfect decoupling and no inter-antenna interference. The group property was important because the detector operations preserved the quaternionic structure of the STBC. For example, the Schur complement operation as defined in Equation (29) of Section 4.2 preserves the quaternionic structure as defined in (14) due to the multiplicative group property observed in Lemma 4.1. This allowed both simple decoding for the individual users by maintaining the structure of the equivalent channel. Therefore, the question here is whether we can do another operation that would ensure such a property in the finite block length case as well.

It turns out that there is a simple way to do this using a technique developed in [15]. This is done by processing the  $(N + \nu)$ -dimensional vectors defined in (7)-(8) as shown in (10).

Let us define the set  $\mathcal{Q}^{(c)}$  of invertible  $2(N + \nu)$ -dimensional square matrices of the form of  $\mathbf{H}_1^{(c)}$  given in (10), i.e.,  $\mathcal{Q}^{(c)}$  is set of  $2 \times 2$  block quaternionic matrices of the special form as in (10) where each block is a circulant matrix. The set  $\mathcal{Q}^{(c)}$  has a multiplicative group property similar to that of the  $2 \times 2$  D-domain matrices  $\mathcal{Q}$  given in Lemma 4.1.

<sup>10</sup>In general, the multiplication of two Toeplitz matrices is not Toeplitz. However, in our case, it turns out to be Toeplitz because of the fully-windowed triangular structure of  $\mathbf{H}_{1,1}^{\hat{z},s}$  and  $\mathbf{H}_{1,2}^{\hat{z},s}$ .

<sup>11</sup>A good measure for the energy in a matrix is its Frobenius norm defined (for a matrix  $\mathbf{A}$ ) as  $\|\mathbf{A}\|_F \stackrel{def}{=} \sqrt{\text{trace}(\mathbf{A}\mathbf{A})}$ .

**Lemma 5.1**  $\mathcal{Q}^{(c)}$  forms a multiplicative group, i.e., they have the following properties

$$\begin{aligned} \text{For } \mathbf{V}_1, \mathbf{V}_2 \in \mathcal{Q}^{(c)}, \mathbf{V}_1 \mathbf{V}_2 \in \mathcal{Q}^{(c)}. \\ \mathbf{V}^{-1} = [\mathbf{I}_2 \otimes (\mathbf{V}_{1,1} \bar{\mathbf{V}}_{1,1} + \mathbf{V}_{1,2} \bar{\mathbf{V}}_{1,2})^{-1}] \bar{\mathbf{V}} \in \mathcal{Q}^{(c)}, \end{aligned} \quad (46)$$

where  $\otimes$  denotes the Kronecker product,

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ -\bar{\mathbf{V}}_{1,2} & \bar{\mathbf{V}}_{1,1} \end{bmatrix}, \quad \bar{\mathbf{V}} = \begin{bmatrix} \bar{\mathbf{V}}_{1,1} & -\mathbf{V}_{1,2} \\ \bar{\mathbf{V}}_{1,2} & \mathbf{V}_{1,1} \end{bmatrix}, \quad (47)$$

with  $\mathbf{V}_{1,1}, \mathbf{V}_{1,2}$  being circulant matrices.

As mentioned earlier, we can parallel all the properties developed in Section 4.1 by observing that a D-domain scalar polynomial is algebraically equivalent to a circulant matrix. Using these properties, we can develop the finite-block joint equalization and interference suppression just as we did with the D-domain form in Sections 4.2 and 4.3. In particular, in parallel to (5), we obtain for two receive antennas and two users for vector  $\mathbf{r}$  of size  $4(N + \nu)$  (i.e., processing 2 receive antennas over two transmission sub-blocks)

$$\mathbf{r}^{(c)} = \begin{bmatrix} \mathbf{r}_1^{(c)} \\ \mathbf{r}_2^{(c)} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{G}_1 \\ \mathbf{H}_2 & \mathbf{G}_2 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{s}} \end{bmatrix} + \begin{bmatrix} \eta_1^{(c)} \\ \eta_2^{(c)} \end{bmatrix}, \quad (48)$$

where  $\tilde{\mathbf{c}}, \tilde{\mathbf{s}}$  are the zero-stuffed  $2(N + \nu)$ -dimensional data vectors as defined from (10). Applying the following zero-forcing matrix  $\mathbf{W}$

$$\mathbf{W} = \begin{bmatrix} \mathbf{I}_{2(N+\nu)} & -\mathbf{G}_1 \mathbf{G}_2^{-1} \\ -\mathbf{H}_2 \mathbf{H}_1^{-1} & \mathbf{I}_{2(N+\nu)} \end{bmatrix}, \quad (49)$$

to (48) would yield

$$\mathbf{W} \mathbf{r}^{(c)} = \tilde{\mathbf{r}}^{(c)} = \begin{bmatrix} \tilde{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}} \\ \tilde{\mathbf{s}} \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1^{(c)} \\ \tilde{\eta}_2^{(c)} \end{bmatrix}. \quad (50)$$

The Schur-complement matrices  $\tilde{\mathbf{H}}$  and  $\tilde{\mathbf{G}}$  have the form

$$\begin{aligned} \tilde{\mathbf{H}} &= \mathbf{H}_1 - \mathbf{G}_1 \mathbf{G}_2^{-1} \mathbf{H}_2 \\ \tilde{\mathbf{G}} &= \mathbf{G}_2 - \mathbf{H}_2 \mathbf{H}_1^{-1} \mathbf{G}_1, \end{aligned} \quad (51)$$

which due to Lemma 5.1 still belong to  $\mathcal{Q}^{(c)}$  and hence this preserves the decoupling property. The whitening filter in this case for stream  $\tilde{\mathbf{c}}$  is

$$\mathbf{U} = [\mathbf{I}_2 \otimes (\mathbf{I}_{N+\nu} + \mathbf{G}_{1,eq}^{(c)} \mathbf{G}_{2,eq}^{(c)-1})^{-\frac{1}{2}}], \quad (52)$$

where  $\mathbf{G}_{k,eq}^{(c)} = \mathbf{G}_{k,1}^{(c)} \tilde{\mathbf{G}}_{k,1}^{(c)} + \mathbf{G}_{k,2}^{(c)} \tilde{\mathbf{G}}_{k,2}^{(c)}$ ,  $k = 1, 2$  and we take the Hermitian square root of Hermitian positive-definite circulant matrices. Given this whitening filter, again we can show as in the D-domain processing case, that  $\tilde{\mathbf{H}} = \mathbf{U}\tilde{\mathbf{H}} \in \mathcal{Q}^{(c)}$  still retains the algebraic properties of the equivalent channel. Also note that since  $\tilde{\mathbf{c}}, \tilde{\mathbf{s}}$  are zero padded sequences, in actual fact for detection, we can work with the tall Toeplitz matrices derived from  $\tilde{\mathbf{H}}, \tilde{\mathbf{G}}$  by removing the columns corresponding to the zero-stuffing. This allows us to obtain linear convolution between the equivalent channel and the data sequences. Hence, any standard technique to detect symbols in ISI channels can then be used.

The discussion about MMSE receivers also proceeds along the same lines as in Section 4.3. However, the difference is that the maximum-likelihood decoding of (10) is more computationally complicated.

We conclude this section with a brief discussion on how to extend the interference cancellation technique to the case of  $K > 2$  users and  $M_r = K$  receive antennas. We consider the finite block length case but the approach applies directly to the  $D$ -domain framework of Section 4.

With  $K$  users, and  $M_r = K$  receive antennas Equation (11) becomes

$$\begin{aligned} \begin{bmatrix} \mathbf{r}_1^{(c)} \\ \mathbf{r}_2^{(c)} \\ \vdots \\ \mathbf{r}_K^{(c)} \end{bmatrix} &= \begin{bmatrix} \mathbf{H}_1^{(1)} & \cdots & \mathbf{H}_1^{(K)} \\ \mathbf{H}_2^{(1)} & \cdots & \mathbf{H}_2^{(K)} \\ \vdots & \cdots & \vdots \\ \mathbf{H}_K^{(1)} & \cdots & \mathbf{H}_K^{(K)} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 \\ \tilde{\mathbf{c}}_2 \\ \vdots \\ \tilde{\mathbf{c}}_K \end{bmatrix} + \begin{bmatrix} \eta_1^{(c)} \\ \eta_2^{(c)} \\ \vdots \\ \eta_K^{(c)} \end{bmatrix} \\ &\stackrel{def}{=} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_1 \\ \vdots \\ \tilde{\mathbf{c}}_{K-1} \end{bmatrix} \\ \tilde{\mathbf{c}}_K \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \eta_1^{(c)} \\ \vdots \\ \eta_{K-1}^{(c)} \end{bmatrix} \\ \eta_K^{(c)} \end{bmatrix}, \end{aligned} \quad (53)$$

where  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{D}$  denote the  $2(K-1)(N+\nu)$ -dimensional square upper-left, the  $2(K-1)(N+\nu) \times 2(N+\nu)$  upper-right, the  $2(N+\nu) \times 2(K-1)(N+\nu)$  lower-left, and the  $2(N+\nu)$ -dimensional square lower right submatrices of the  $K$ -user channel matrix in (53).

Applying the linear decorrelating matrix filter  $\mathbf{W} = \begin{bmatrix} \mathbf{I}_{2(K-1)(N+\nu)} & -\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{C}\mathbf{A}^{-1} & \mathbf{I}_{2(N+\nu)} \end{bmatrix}$  to (53), we get

$$\begin{bmatrix} \begin{bmatrix} \tilde{\mathbf{r}}_1^{(c)} \\ \vdots \\ \tilde{\mathbf{r}}_{K-1}^{(c)} \end{bmatrix} \\ \tilde{\mathbf{r}}_K^{(c)} \end{bmatrix} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) \begin{bmatrix} \tilde{\mathbf{c}}_1 \\ \vdots \\ \tilde{\mathbf{c}}_{K-1} \end{bmatrix} \\ (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})\tilde{\mathbf{c}}_K \end{bmatrix} + \text{noise}. \quad (54)$$

First, we detect  $\tilde{\mathbf{c}}_K$  from  $\tilde{\mathbf{r}}_K^{(c)}$  with space-time diversity gains due to the fact that  $(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) \in \mathcal{Q}^{(c)}$ . Then, we repeat the above dimension reduction procedure to iteratively detect  $\tilde{\mathbf{c}}_{K-1}, \dots, \tilde{\mathbf{c}}_1$ .



## 6 Conclusion

In this paper we presented a space-time combined interference suppression, equalization, and decoding scheme for multiple synchronous ISI multiple access channel, with each user equipped with multiple transmit antennas. We demonstrated that the diversity order of  $2M_r(\nu + 1)$  is achievable when optimum ML decoding is applied. This quantifies the increase in diversity order, at the same transmission rate, for *each* user using multiple antennas in an ISI-MAC. For most of the paper we illustrated the techniques using  $M_r = 2$  receive antennas and  $K = 2$  users, though the techniques can be easily extended to arbitrary  $K$  and  $M_r$ . We developed techniques for both perfectly decoupling two users (“zero-forcing”) and using an MMSE algorithm where both crucially utilize the time-reversal space-time coding structure employed by the two users. Therefore, from a network point of view, one can pack multiple users obtain the same performance as a single user system at the cost of higher receiver complexity. On the other hand by lowering the receiver complexity, by using linear detectors, one can still increase spectral efficiency at a slightly deteriorated error performance. We can easily incorporate iterative techniques that build on these basic approaches.

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## A Proof of Theorem 3.2

We give this proof for the case of two users ( $K = 2$ ) and  $M_r = 2$  receive antennas. The steps can be very easily generalized to the case with  $K > 2$  users and  $M_r > 2$  receive antennas. The idea of the proof is that the pairwise error probability (PEP) of the ML decoder can be derived in terms of the error vectors of the different users of the multiple-access channel. By using the derived expression of the PEP, we can show that the diversity order is achievable. The proof relies quite heavily on the quaternionic structure that the TR-STBC imposes on the equivalent channels in (11).

We begin with the following well-known observation [7] on circulant matrices.

**Fact A.1** *A circulant matrix  $\mathbf{C}$  of size  $M$  has an eigendecomposition  $\mathbf{C} = \mathbf{Q}\mathbf{\Lambda}\bar{\mathbf{Q}}$  with its eigenvectors as the Fourier matrix  $\mathbf{Q}$  whose elements are given by  $[\mathbf{Q}]_{p,q} = \exp(-j\frac{2\pi}{M}(p-1)(q-1))$ . Moreover,  $\mathbf{\Lambda} = \text{diag}(\mathbf{Q}\mathbf{c}_1)$ , where  $\mathbf{c}_1$  is the first column of  $\mathbf{C}$ , and  $\text{diag}(\cdot)$  creates a diagonal matrix from the elements of a vector.*

Using this fact, we can represent (11) (for the two-user case) in the frequency domain as

$$\mathbf{R} = \begin{bmatrix} \mathbf{\Lambda}_{H_1} & \mathbf{\Lambda}_{G_1} \\ \mathbf{\Lambda}_{H_2} & \mathbf{\Lambda}_{G_2} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{C}} \\ \tilde{\mathbf{S}} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\eta}} \\ \tilde{\boldsymbol{\eta}} \end{bmatrix}, \quad (55)$$

where  $\mathbf{R} = [\mathbf{R}_1^T, \mathbf{R}_2^T]^T$ , with  $\mathbf{R}_k = \check{\mathbf{Q}}\mathbf{r}_k$  and

$$\check{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{bmatrix}. \quad (56)$$

Furthermore,

$$\mathbf{\Lambda}_{H_k} = \begin{bmatrix} \mathbf{\Lambda}_{H_{k,1}} & \mathbf{\Lambda}_{H_{k,2}} \\ -\bar{\mathbf{\Lambda}}_{H_{k,2}} & \bar{\mathbf{\Lambda}}_{H_{k,1}} \end{bmatrix}, \quad (57)$$

with  $\mathbf{\Lambda}_{H_{k,l}}$ ,  $l = 1, 2$  contains the eigenvalues of  $\mathbf{H}_{k,l}^{(1)}$ . Similarly,

$$\mathbf{\Lambda}_{G_k} = \begin{bmatrix} \mathbf{\Lambda}_{G_{k,1}} & \mathbf{\Lambda}_{G_{k,2}} \\ -\bar{\mathbf{\Lambda}}_{G_{k,2}} & \bar{\mathbf{\Lambda}}_{G_{k,1}} \end{bmatrix}, \quad (58)$$

with  $\mathbf{\Lambda}_{G_{k,l}}$ ,  $l = 1, 2$  contains the eigenvalues of  $\mathbf{H}_{k,l}^{(2)}$ . The data vectors are written as  $\tilde{\mathbf{C}} = \check{\mathbf{Q}}\mathbf{c}^{(1)}$  and  $\tilde{\mathbf{S}} = \check{\mathbf{Q}}\mathbf{c}^{(2)}$ . Given that the Fourier transformation is orthonormal,  $\eta$  is still white Gaussian noise with variance 1.

The PEP for (55) can be bounded as [13],

$$\mathbb{P}(\mathbf{c} \rightarrow \mathbf{c}' | \mathbf{H}^{(1)}, \mathbf{H}^{(2)}) \leq \exp(-d^2(\mathbf{R}, \mathbf{R}')/4), \quad (59)$$

where

$$d^2(\mathbf{R}, \mathbf{R}') = \begin{bmatrix} \bar{\mathbf{e}}_c & \bar{\mathbf{e}}_s \end{bmatrix} \begin{bmatrix} \bar{\mathbf{\Lambda}}_{H_1} & \bar{\mathbf{\Lambda}}_{H_2} \\ \bar{\mathbf{\Lambda}}_{G_2} & \bar{\mathbf{\Lambda}}_{G_1} \end{bmatrix} \begin{bmatrix} \mathbf{\Lambda}_{H_1} & \mathbf{\Lambda}_{G_1} \\ \mathbf{\Lambda}_{H_2} & \mathbf{\Lambda}_{G_2} \end{bmatrix} \begin{bmatrix} \mathbf{e}_c \\ \mathbf{e}_s \end{bmatrix}, \quad (60)$$

and  $\mathbf{e}_c = \mathbf{c}^{(1)} - \mathbf{c}^{(1)'} = [\mathbf{e}_{c_1}^T, \mathbf{e}_{c_2}^T]^T$ . After re-arranging terms, we can rewrite the quadratic form in (60) as

$$d^2(\mathbf{R}, \mathbf{R}') = \begin{bmatrix} \bar{\lambda}_{H_1} & \bar{\lambda}_{H_2} & \bar{\lambda}_{G_1} & \bar{\lambda}_{G_2} \end{bmatrix} \begin{bmatrix} \mathbf{I}_2 \otimes (\mathbf{E}_c \bar{\mathbf{E}}_c) & \mathbf{I}_2 \otimes (\mathbf{E}_c \bar{\mathbf{E}}_s) \\ \mathbf{I}_2 \otimes (\mathbf{E}_s \bar{\mathbf{E}}_c) & \mathbf{I}_2 \otimes (\mathbf{E}_s \bar{\mathbf{E}}_s) \end{bmatrix} \begin{bmatrix} \lambda_{H_1} \\ \lambda_{H_2} \\ \lambda_{G_1} \\ \lambda_{G_2} \end{bmatrix}, \quad (61)$$

where  $\bar{\lambda}_{H_k} = [\bar{\lambda}_{H_{k,1}}, \bar{\lambda}_{H_{k,2}}]$ ,  $\lambda_{H_{k,1}} = \text{vec}(\mathbf{\Lambda}_{H_{k,1}})$ , and  $\bar{\lambda}_{G_k}$  is defined similarly. Applying the  $\text{vec}(\cdot)$  operation to a diagonal matrix constructs a vector from its diagonal elements. Also

$$\mathbf{E}_c = \begin{bmatrix} \mathbf{E}_{c_1} & -\bar{\mathbf{E}}_{c_2} \\ \mathbf{E}_{c_2} & \bar{\mathbf{E}}_{c_1} \end{bmatrix}, \quad \mathbf{E}_s = \begin{bmatrix} \mathbf{E}_{s_1} & -\bar{\mathbf{E}}_{s_2} \\ \mathbf{E}_{s_2} & \bar{\mathbf{E}}_{s_1}^* \end{bmatrix}, \quad (62)$$

with  $\mathbf{E}_{c_k} = \text{diag}(\mathbf{e}_{c_k})$ ,  $\mathbf{E}_{s_k} = \text{diag}(\mathbf{e}_{s_k})$ ,  $k = 1, 2$ . Using Fact A.1, we can write  $\lambda_{H_k} = \check{\mathbf{Q}}\mathbf{h}_k$ , with  $\mathbf{h}_k = [\mathbf{h}_{k,1}^T, \mathbf{h}_{k,1}^T]^T$  being the first columns of the circulant matrices  $\mathbf{H}_{k,l}^{(1)}$ . Since we have FIR channels, only the first  $\nu + 1$  elements of  $\mathbf{h}_{k,1}$  are non-zero, and we denote this by the length- $(\nu + 1)$  vector  $\tilde{\mathbf{h}}_{k,1}$  and form  $\tilde{\mathbf{h}}_k = [\tilde{\mathbf{h}}_{k,1}^T, \tilde{\mathbf{h}}_{k,2}^T]^T$ . The definition for  $\lambda_{G_k}$  is identical and we do not repeat it here. Hence the quadratic form in (61) can be rewritten as

$$d^2(\mathbf{R}, \mathbf{R}') = \begin{bmatrix} \tilde{\mathbf{h}}_1 & \tilde{\mathbf{h}}_2 & \tilde{\mathbf{g}}_1 & \tilde{\mathbf{g}}_2 \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{I}_2 \otimes (\check{\mathbf{Q}}\mathbf{E}_c\bar{\mathbf{E}}_c\check{\mathbf{Q}}) & \mathbf{I}_2 \otimes (\check{\mathbf{Q}}\mathbf{E}_c\bar{\mathbf{E}}_s\check{\mathbf{Q}}) \\ \mathbf{I}_2 \otimes (\check{\mathbf{Q}}\mathbf{E}_s\bar{\mathbf{E}}_c\check{\mathbf{Q}}) & \mathbf{I}_2 \otimes (\check{\mathbf{Q}}\mathbf{E}_s\bar{\mathbf{E}}_s\check{\mathbf{Q}}) \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \tilde{\mathbf{h}}_1 \\ \tilde{\mathbf{h}}_2 \\ \tilde{\mathbf{g}}_1 \\ \tilde{\mathbf{g}}_2 \end{bmatrix}, \quad (63)$$

where  $\tilde{\mathbf{Q}}$  is a  $2(N + \nu) \times 2(\nu + 1)$  block diagonal matrix with the blocks being the first  $(\nu + 1)$  columns of the Fourier matrix  $\mathbf{Q}$ . Therefore, inserting this in (59) and averaging the Gaussian quadratic form over the channel parameters in a standard manner (see for example [13]), we obtain

$$\mathbb{P}(\mathbf{c} \rightarrow \mathbf{c}') \leq \prod_{l=1}^{8(\nu+1)} \frac{1}{1 + \frac{\gamma(l)SNR}{4}}, \quad (64)$$

where  $\gamma(l)$  represents the eigenvalues of  $\mathcal{E}$  in the quadratic form of (63). Hence the diversity order depends on the rank of  $\mathcal{E}$ , and we next show that this matrix has rank  $4(\nu + 1)$  for any error sequence on each user yielding the diversity order result claimed in Theorem 3.2.

First note that in (63), since we are examining the diversity order for any user, the maximal rank of  $\mathcal{E}$  is  $4(\nu + 1)$ . This is true because we can set, for example,  $\mathbf{E}_c \neq 0, \mathbf{E}_s = 0$  when there is an error only in the first user's transmission. The main step is to show that this maximal rank is achieved and therefore giving us the result claimed in Theorem 3.2. Observe that by row and column interchanges of  $\mathcal{E}$ , it can be written as

$$\mathbf{I}_2 \otimes \begin{bmatrix} \bar{\mathbf{Q}}\mathbf{E}_c\bar{\mathbf{E}}_c\tilde{\mathbf{Q}} & \bar{\mathbf{Q}}\mathbf{E}_c\bar{\mathbf{E}}_s\tilde{\mathbf{Q}} \\ \bar{\mathbf{Q}}\mathbf{E}_s\bar{\mathbf{E}}_c\tilde{\mathbf{Q}} & \bar{\mathbf{Q}}\mathbf{E}_s\bar{\mathbf{E}}_s\tilde{\mathbf{Q}} \end{bmatrix} = \mathbf{I}_2 \otimes \left( \begin{bmatrix} \bar{\mathbf{Q}}\mathbf{E}_c \\ \bar{\mathbf{Q}}\mathbf{E}_s \end{bmatrix} \begin{bmatrix} \bar{\mathbf{E}}_c\tilde{\mathbf{Q}} & \bar{\mathbf{E}}_s\tilde{\mathbf{Q}} \end{bmatrix} \right). \quad (65)$$

Next we will show that  $\bar{\mathbf{Q}}\mathbf{E}_c$  has rank  $2(\nu + 1)$  if  $\mathbf{E}_c \neq 0$ . This will prove that  $\mathcal{E}$  achieves a rank of  $4(\nu + 1)$ . Let the  $(N + \nu) \times (\nu + 1)$  matrix containing the first  $(\nu + 1)$  columns of  $\mathbf{Q}$  be denoted by  $\mathbf{T}_Q$  and  $\bar{\mathbf{T}}_Q = [\psi_1, \dots, \psi_{N+\nu}]$  with  $[\psi_k]_p = e^{j\frac{2\pi}{N+\nu}(k-1)(p-1)}$ ,  $k = 1, \dots, N + \nu$ ,  $p = 1, \dots, \nu + 1$ . Since  $\mathbf{T}_Q$  is the truncated Fourier matrix it has full rank of  $(\nu + 1)$  and hence any  $(\nu + 1)$  rows of  $\mathbf{T}_Q$  are linearly independent. This therefore allows to write

$$\bar{\mathbf{Q}}\mathbf{E}_c = \begin{bmatrix} \bar{\mathbf{T}}_Q\mathbf{E}_{c_1} & -\bar{\mathbf{T}}_Q\bar{\mathbf{E}}_{c_2} \\ \bar{\mathbf{T}}_Q\mathbf{E}_{c_2} & \bar{\mathbf{T}}_Q\bar{\mathbf{E}}_{c_1} \end{bmatrix}. \quad (66)$$

It is easy to show that if  $\mathbf{c}_k - \mathbf{c}'_k \neq 0$ ,  $k = 1, 2$  then the diagonal matrix  $\mathbf{E}_{c_k}$  has at least  $(\nu + 1)$  non-zero entries using the property that  $\mathbf{Q}\mathbf{I}_{z_s}$  has full rank<sup>12</sup>. Without loss of generality we can assume  $\mathbf{c}_1 - \mathbf{c}'_1 \neq 0$ , then  $\mathbf{E}_{c_1}$  has at least  $(\nu + 1)$  non-zero entries and let us choose the  $2(\nu + 1)$  columns corresponding to those entries in (66), and denote this set of columns by  $f(1), \dots, f(\nu + 1)$ . If these columns are linearly dependent, then there exist scalars  $\{\alpha_l, \beta_l\}_{l=1}^{\nu+1}$  not all zero such that

$$\sum_{l=1}^{\nu+1} \left\{ \alpha_l \begin{bmatrix} \mathbf{e}_{c_1}(f(l))\psi_{f(l)} \\ \mathbf{e}_{c_2}(f(l))\psi_{f(l)} \end{bmatrix} + \beta_l \begin{bmatrix} -\bar{\mathbf{e}}_{c_2}(f(l))\psi_{f(l)} \\ \bar{\mathbf{e}}_{c_1}(f(l))\psi_{f(l)} \end{bmatrix} \right\} = \mathbf{0}_{2(N+\nu) \times 1}. \quad (67)$$

Since  $\mathbf{T}_Q$  is full rank of  $\nu + 1$  this reduces to the simpler set of equations

$$\begin{bmatrix} \tilde{\mathbf{e}}_{c_1}(f(l)) & -\tilde{\mathbf{e}}_{c_2}(f(l)) \\ \tilde{\mathbf{e}}_{c_2}(f(l)) & \tilde{\mathbf{e}}_{c_1}(f(l)) \end{bmatrix} \begin{bmatrix} \alpha_l \\ \beta_l \end{bmatrix} = \mathbf{0}_{2 \times 1}, \quad l = 1, \dots, \nu + 1 \quad (68)$$

<sup>12</sup>Remember that  $\mathbf{e}_{c_k} = \mathbf{Q}\mathbf{I}_{z_s}(\mathbf{c}_k - \mathbf{c}'_k)$ ,  $k = 1, 2$  and  $\mathbf{E}_{c_k} = \text{diag}(\mathbf{e}_{c_k})$ .

which cannot be true since we have chosen  $f(l)$  such that  $\tilde{\mathbf{e}}_{c_1}(f(l)) \neq 0, \forall l$ . Hence, there do not exist scalars  $\{\alpha_l, \beta_l\}_{l=1}^{\nu+1}$  not all zero such that (67) holds. This proves that  $\bar{\mathbf{Q}}\mathbf{E}_c$  has rank  $2(\nu+1)$ . To conclude the proof of Theorem 3.2 for  $K=2, M_r=2$ , we can use the standard union bound argument to bound the average probability in terms of the pairwise error probability. Since we are using a constant rate code and the rank of  $\mathcal{E}$  is  $4(\nu+1)$ , the PEP decays at a rate  $\frac{1}{SNR^{4(\nu+1)}}$ , therefore the diversity order according to Definition 3.1 is  $4(\nu+1)$ .

For the general case, the proof structure is identical. It can be easily verified that the equivalent  $M_r K(N+\nu)$  square matrix  $\mathcal{E}$  in (63) has the  $(p, q)$ th square block components as  $\mathbf{I}_{M_r} \otimes (\mathbf{E}_{\mathbf{c}(p)} \bar{\mathbf{E}}_{\mathbf{c}(q)})$  of size  $M_r 2(N+\nu)$ . The rank of  $\mathcal{E}$  would again be determined by the rank of  $\bar{\mathbf{Q}}\mathbf{E}_c$  which is found by the above argument to be  $2(\nu+1)$ , yielding the rank of  $\mathcal{E}$  as  $2M_r(\nu+1)$ . This will allow us to show that the diversity order is  $2M_r(\nu+1)$ .

## B Proof of Theorem 4.8

**Proof:** We use induction, for  $k=0, 1$ , this is clearly true due to Lemmas 4.4, 4.7. By the inductive hypothesis, for  $0 \leq i < l$  we have

$$\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^i(D)\mathbf{v}_2^v(D) = 0 = \bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}^i(D)\mathbf{x}_2^v(D). \quad (69)$$

Continuing the induction we have

$$\begin{aligned} \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^l(D)\mathbf{v}_2^v(D) &= \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^{l-1}(D) \left\{ \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} + \frac{1}{\gamma}\mathbf{I}_4 \right\} \mathbf{v}_2^v(D) \\ &\stackrel{(a)}{=} \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^{l-1}(D) \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D). \end{aligned} \quad (70)$$

The equality in (a) holds because we can show that for

$$\mathbf{y}^v(D) \stackrel{def}{=} \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D), \quad (71)$$

$\mathbf{y}^v(D) \in \mathcal{H}$  and  $\langle \mathbf{v}_1^v(D), \mathbf{y}^v(D) \rangle = 0$  and hence  $\{\mathbf{v}_1^v(D), \mathbf{y}^v(D)\}$  form a *special pair* for  $\mathcal{H}$  and hence by invoking the inductive hypothesis (69) we get (71). Next, note that

$$\mathbf{z}_i^v(D) \stackrel{def}{=} \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_i^v(D) \in \mathcal{G} \quad (72)$$

and it can easily be verified that  $\{\mathbf{z}_1^v(D), \mathbf{z}_2^v(D)\}$  forms a *special pair* for  $\mathcal{G}$ . Therefore, by the inductive hypothesis

$$\bar{\mathbf{z}}_1^v(\bar{D}^{-1})\mathbf{M}^{l-2}(D)\mathbf{z}_2^v(D) = 0. \quad (73)$$

Now we can rewrite (70) as

$$\begin{aligned} \bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^l(D)\mathbf{v}_2^v(D) &= \bar{\mathbf{v}}_1^v(\bar{D}^{-1}) \left\{ \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \right. \\ &\left. + \frac{1}{\gamma}\mathbf{I}_4 \right\} \mathbf{M}^{l-2}(D) \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D) \end{aligned} \quad (74)$$

$$\stackrel{(a)}{=} \bar{\mathbf{v}}_1^v(\bar{D}^{-1}) \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{M}^{l-2}(D) \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D), \quad (75)$$

where (a) is due to the inductive hypothesis and (73). Now since,

$$\mathbf{y}_1^v(D) \stackrel{def}{=} \begin{bmatrix} \mathbf{H}_1(D) \\ \mathbf{H}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{H}}_1(\bar{D}^{-1}) & \bar{\mathbf{H}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_1^v(D) \in \mathcal{H} \quad (76)$$

and  $\{\mathbf{y}_1^v(D), \mathbf{v}_2^v(D)\}$  forms a *special pair* for  $\mathcal{H}$ , we see that the last line in (74) is of the form  $\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^{l-1}(D)\mathbf{v}_2^v(D)$ . Therefore, by iterating in  $l$  and using Lemma 4.5 we obtain

$$\bar{\mathbf{v}}_1^v(\bar{D}^{-1})\mathbf{M}^l(D)\mathbf{v}_2^v(D) = \bar{\mathbf{v}}_1^v(\bar{D}^{-1}) \begin{bmatrix} \mathbf{G}_1(D) \\ \mathbf{G}_2(D) \end{bmatrix} \begin{bmatrix} \bar{\mathbf{G}}_1(\bar{D}^{-1}) & \bar{\mathbf{G}}_2(\bar{D}^{-1}) \end{bmatrix} \mathbf{v}_2^v(D) = 0. \quad (77)$$

This completes the inductive proof and the proof for  $\bar{\mathbf{x}}_1^v(\bar{D}^{-1})\mathbf{M}^k(D)\mathbf{x}_2^v(D) = 0$  is identical.  $\blacksquare$

## C Derivation Of Equation (44)

In this appendix, we show that multiplying  $\mathbf{r}_1$  in (9) by  $\bar{\mathbf{H}}_1$  decouples  $\mathbf{c}_1$  and  $\mathbf{c}_2$  while keeping the two noise components uncorrelated. Starting from (9), we have

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} \bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,1}^{zs} + \mathbf{J}_N (\mathbf{H}_{1,2}^{zs})^T \mathbf{H}_{1,2}^{*zs} \mathbf{J}_N & \bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,2}^{zs} - \mathbf{J}_N (\mathbf{H}_{1,2}^{zs})^T \mathbf{H}_{1,1}^{*zs} \mathbf{J}_N \\ \bar{\mathbf{H}}_{1,2}^{zs} \mathbf{H}_{1,1}^{zs} - \mathbf{J}_N (\mathbf{H}_{1,1}^{zs})^T \mathbf{H}_{1,2}^{*zs} \mathbf{J}_N & \bar{\mathbf{H}}_{1,2}^{zs} \mathbf{H}_{1,2}^{zs} + \mathbf{J}_N (\mathbf{H}_{1,1}^{zs})^T \mathbf{H}_{1,1}^{*zs} \mathbf{J}_N \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \bar{\mathbf{H}}_1 \begin{bmatrix} \eta_1 \\ \mathbf{J}_{N+\nu} \eta_2 \end{bmatrix} \\ &= \begin{bmatrix} \bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,1}^{zs} + \bar{\mathbf{H}}_{1,2}^{zs} \mathbf{H}_{1,2}^{zs} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,1}^{zs} + \bar{\mathbf{H}}_{1,2}^{zs} \mathbf{H}_{1,2}^{zs} \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{bmatrix} + \begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix}. \end{aligned}$$

To prove the second equality above, we only need to show that

$$\bar{\mathbf{H}}_{1,1}^{zs} \mathbf{H}_{1,2}^{zs} = \mathbf{J}_N (\mathbf{H}_{1,2}^{zs})^T \mathbf{H}_{1,1}^{*zs} \mathbf{J}_N. \quad (78)$$

It will be convenient in the proof to work with the *circulant* versions of the matrices  $\mathbf{H}_{1,1}$  and  $\mathbf{H}_{1,2}$  which are obtained by wrapping around their last  $\nu$  columns and will be denoted by  $\mathbf{H}_{1,1}^c$  and  $\mathbf{H}_{1,2}^c$ ,

respectively. Then, it immediately follows that <sup>13</sup>

$$\begin{aligned}\mathbf{H}_{1,1}^{zs} &= \mathbf{H}_{1,1}\mathbf{I}_{zs} = \mathbf{H}_{1,1}^c\mathbf{I}_{zs} \\ \mathbf{H}_{1,2}^{zs} &= \mathbf{H}_{1,2}\mathbf{I}_{zs} = \mathbf{H}_{1,2}^c\mathbf{I}_{zs} .\end{aligned}\tag{79}$$

$$\tag{80}$$

Starting from the right-hand side of (78), we have the following equalities

$$\mathbf{J}_N(\mathbf{H}_{1,2}^{zs})^T\mathbf{H}_{1,1}^{*zs}\mathbf{J}_N = \mathbf{J}_N\mathbf{I}_{zs}^T\mathbf{H}_{1,2}^c\mathbf{H}_{1,1}^{*c}\mathbf{I}_{zs}\mathbf{J}_N\tag{81}$$

$$= \mathbf{J}_N\mathbf{I}_{zs}^T\mathbf{Q}^*\mathbf{\Lambda}_{1,2}^T\mathbf{Q}^T\mathbf{Q}^*\mathbf{\Lambda}_{1,1}^*\mathbf{Q}^T\mathbf{I}_{zs}\mathbf{J}_N\tag{82}$$

$$= \mathbf{J}_N\mathbf{I}_{zs}^T\mathbf{J}_{N+\nu}\mathbf{Q}\mathbf{\Lambda}_{1,1}^*\mathbf{\Lambda}_{1,2}\bar{\mathbf{Q}}\mathbf{J}_{N+\nu}\mathbf{I}_{zs}\mathbf{J}_N\tag{83}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{I}_N \end{bmatrix} \bar{\mathbf{H}}_{1,2}^c\mathbf{H}_{2,c} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_N \end{bmatrix}\tag{84}$$

$$= \bar{\mathbf{I}}_{zs}\bar{\mathbf{H}}_{1,1}^c\mathbf{H}_{1,2}^c\mathbf{I}_{zs}\tag{85}$$

$$= \bar{\mathbf{H}}_{1,1}^{zs}\mathbf{H}_{1,2}^{zs},\tag{86}$$

where  $\mathbf{Q}$  is the FFT matrix. Note that (81) and (86) follow from (79) and that (82) holds because circulant matrices are diagonalizable by the FFT matrix. The equality in (83) uses the property that pre- and post-multiplication of a circulant matrix by the reversal matrix yields the transpose of the circulant matrix. Continuing, Equation (84) uses the matrix identities  $\mathbf{J}_N\mathbf{I}_{zs}^T\mathbf{J}_{N+\nu} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_N \end{bmatrix}$  and  $\mathbf{J}_{N+\nu}\mathbf{J}_{N+\nu} = \mathbf{I}_{N+\nu}$ . Finally, (85) uses the fact that for any circulant matrix  $\mathbf{A}_c$ , we have  $\begin{bmatrix} \mathbf{I}_N & \mathbf{0} \end{bmatrix} \mathbf{A}_c \begin{bmatrix} \mathbf{I}_N \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_N \end{bmatrix} \mathbf{A}_c \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_N \end{bmatrix}$ . In words, this property states that the first  $N \times N$  submatrix of a circulant matrix is identical to its last  $N \times N$  submatrix which follows directly from the circulant structure.

We conclude by showing that the two noise components remain uncorrelated after application of the matched filter  $\bar{\mathbf{H}}_1$ . From (9), the auto-correlation matrix of the filtered noise is given by

$$\begin{aligned}E\left[\begin{bmatrix} \tilde{\eta}_1 \\ \tilde{\eta}_2 \end{bmatrix}\right]\left[\begin{bmatrix} \tilde{\eta}_1 & \tilde{\eta}_2 \end{bmatrix}\right] &= \bar{\mathbf{H}}_1 \begin{bmatrix} \eta_1 \\ \mathbf{J}_{N+\nu}\eta_2 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_1 & \tilde{\eta}_2\mathbf{J}_{N+\nu} \end{bmatrix} \mathbf{H}_1 \\ &= \bar{\mathbf{H}}_1 \begin{bmatrix} \mathbf{I}_{N+\nu} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{N+\nu}\mathbf{J}_{N+\nu} \end{bmatrix} \mathbf{H}_1 \\ &= \bar{\mathbf{H}}_1\mathbf{H}_1 = \begin{bmatrix} \bar{\mathbf{H}}_{1,1}^{zs}\mathbf{H}_{1,1}^{zs} + \bar{\mathbf{H}}_{1,2}^{zs}\mathbf{H}_{1,2}^{zs} & \mathbf{0}_{N \times N} \\ \mathbf{0}_{N \times N} & \bar{\mathbf{H}}_{1,1}^{zs}\mathbf{H}_{1,1}^{zs} + \bar{\mathbf{H}}_{1,2}^{zs}\mathbf{H}_{1,2}^{zs} \end{bmatrix} .\end{aligned}$$

<sup>13</sup>These relations hold because multiplying by  $\mathbf{I}_{zs}$  makes any differences in the last  $\nu$  columns irrelevant.

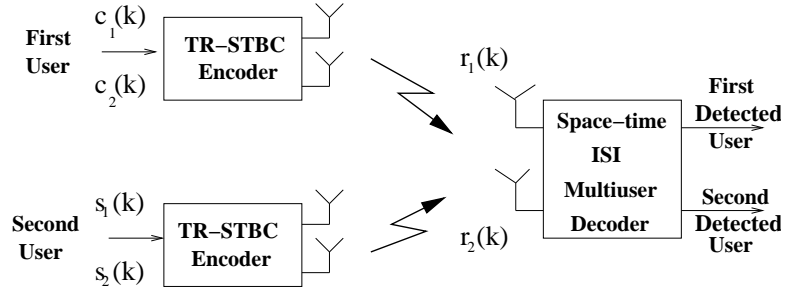


Figure 1: System configuration.

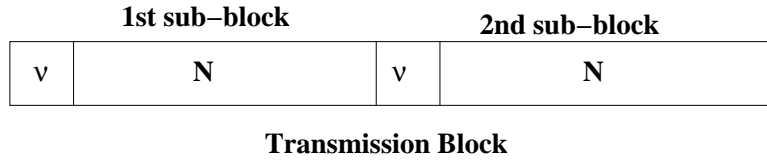


Figure 2: Block Transmission Format

**Transmission block from first antenna**



**Transmission block from second antenna**



**G = Guard Sequence of length  $v$**

Figure 3: Transmission Format for Time-Reversal Space-Time Block Coding



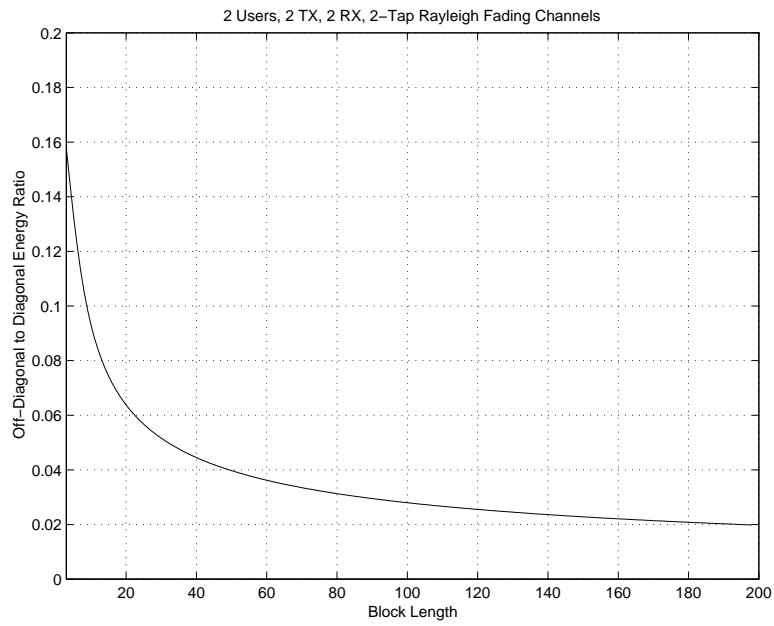


Figure 4: Variation of Inter-Antenna Interference to Signal Ratio with Block Length Without the Circulant Matrix Formulation of (10)