Fundamental Limits of Diversity-Embedded Codes over Fading Channels

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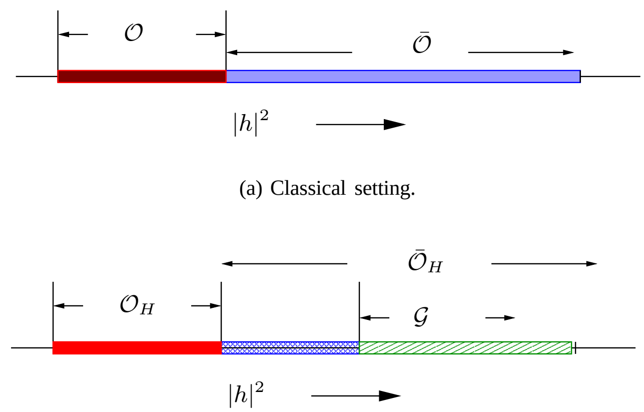
***Abstract*— Diversity-embedded codes for fading channels are high-rate codes that are designed so that they have a high-diversity code embedded within them. This allows a form of communication where the high-rate code opportunistically takes advantage of good channel realizations whereas the embedded high-diversity code ensures that at least part of the information is received reliably. This can also be thought as coding the data into two streams such that the high-priority stream has higher reliability than the low-priority stream. For SISO (single-input-single-output), SIMO, MISO and parallel fading channels, we characterize the achievable rates and reliability of the two streams in the high SNR regime in terms of the diversity-multiplexing tradeoff. We exhibit the performance gain over a single-stream code. We also show some constructions for finite block lengths that achieve the optimal performance.**

I. Introduction

The classical approach towards code design for slow fading channels is to maximize the data rate given a desired level of reliability against deep fades. Clearly, there is a tradeoff between data rate and reliability: the higher the desired level of reliability, the lower the data rate one can achieve.

A different approach was proposed in [3], where it was asked whether there exists a strategy that combines high-rate communications with high reliability (diversity). Clearly the overall code will still be governed by the rate-reliability tradeoff, but the idea is to ensure the high reliability (diversity) of at least part of the total information. These are called *diversity-embedded* codes. Design criteria along with preliminary constructions and bounds were obtained in [3].

This paper addresses the fundamental limits of the performance achievable by such diversity-embedded codes. A natural setting to address this question is the *outage formulation*. The classical outage formulation divides the set of channel realizations into an outage set $\mathcal{O}$ and a non-outage set $\bar{\mathcal{O}}$: it requires that a code has to be designed such that the transmitted message can be decoded with arbitrary small error probability on all the channels in the non-outage set. Since the code must work for all such channels, the data rate is limited by the worst channel in the non-outage set. See Figure I(a). For a given outage probability p, the performance limit is the outage capacity C_p, which is the maximum achievable such worst-case rate, maximum over all non-outage set with probability of at least $1-p$. Inverting the mapping from p to C_p, we get the outage probability $p(R)$ for a given target rate R.



(a) Classical setting.

(b) Diversity embedded setting.

Fig. 1. Outage events in the classical setting and for diversity-embedded coding.

The classical outage formulation is catered for the classical scenario where there is a single stream with a single desired level of reliability. Note that in this scenario, the communication strategy cannot take advantage of the opportunity when the channel happens to be stronger than the worst channel in the non-outage set. Diversity-embedded coding can be modeled by having two messages m_H (high-priority stream) and m_L (low-priority stream) (Figure 2), such that a code has to be designed to ensure that m_H is decoded with arbitrary small error probability whenever the channel is not in outage ($\mathcal{O}_H$) and in addition m_L is to be decoded whenever the channel is in a set $\mathcal{G} \subset \bar{\mathcal{O}}_H$ of good channels. See Figure I(b). The streams m_H and m_L have reliability $p_H = \mathcal{P}(\mathcal{O}_H)$ and $p_L = \mathcal{P}(\bar{\mathcal{G}})$ respectively; the low-priority stream m_L has lower reliability but it represents additional information that can be pumped through the channel when it is good. The performance limit is then characterized by the set of all achievable 4-tuples (R_H, p_H, R_L, p_L), where R_H and R_L are the rates supportable by the worst channels in $\bar{\mathcal{O}}_H$ and $\mathcal{G}$ respectively.

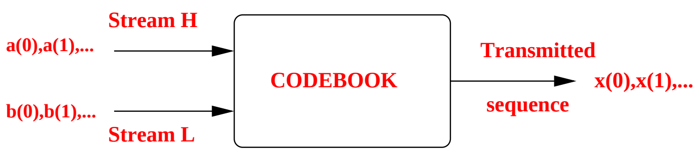


Fig. 2. Embedded codebook

For general SNR's, it is difficult to explicitly characterize the exact outage performance. Instead, we focus on the high SNR regime, where the outage performance can be approximated by a tradeoff between the multiplexing and diversity gains of a code. This *diversity-multiplexing (DM) tradeoff* formulation was introduced in [11] in the single-stream setting, where the optimal diversity gain $d^*(r)$ for a given multiplexing gain is characterized. We extend the formulation to the diversity-embedded setting, where the tradeoff is between the multiplexing and diversity gains r_H, r_L, d_H, d_L of the two streams. For the SISO, MISO, SIMO and parallel Rayleigh fading channels, we compute this tradeoff.

What is the best outage performance achievable in the diversity-embedded setting? Clearly:

$$\begin{aligned} R_H &\leq C_{p_H} \\ R_H + R_L &\leq C_{p_L} \end{aligned}$$

where the second constraint says that the total rate of the two streams we can achieve when the channel is good cannot exceed that of the optimal code designed particularly for the level of reliability p_L. But since we are now designing a code that has to work simultaneously for *both* reliability levels, it is not clear whether these upper bounds can be simultaneously achieved. In fact, it is not difficult to see that, for general SNR's, this cannot be done, even for the SISO channel. However, we show that, in terms of the diversity-multiplexing tradeoff in the high SNR regime, this is in fact possible, as long as the channel has a single degree of freedom (i.e., for the SISO, MISO and SIMO channels). That is, we can achieve simultaneously $(r_H, d^*(r_H), r_L, d^*(r_L + r_H))$. In analogy with rate distortion theory, we call this property *successive refinable*, since we can simultaneously operate at any two points on the DM tradeoff curve. In contrast, we the parallel channel is *not* successively refinable using a Gaussian coding strategy. Nevertheless, we show that, in terms of diversity-multiplexing tradeoff, one can always do better using a diversity-embedded code than a single stream code.

One can view diversity-embedded codes as *broadcast codes* if one identifies a channel realization with a user: one wants to deliver a *common* message m_H to all users in $\bar{\mathcal{O}}$ and a *private* message m_L to all users in $\mathcal{G}$. This point of view was in fact proposed by Cover [2] in his original broadcast paper. Shamai [7] applied this approach to the SISO fading channel but his performance objective was the transmission rate averaged over the channels rather than the outage probabilities. This approach was further explored in [8] which extended the average rate maximization problem to multiple antenna channels. In all these works, superposition coding and successive decoding were used as an achievable strategy: the codeword for the common message m_H is superimposed on that for the private message m_L, and is decoded first. While this strategy is clearly optimal for the SISO channel since the resulting broadcast channel is degraded, we will see that it is in fact DM tradeoff-optimal for the MISO and SIMO cases as well.

This paper is organized as follows. In Section II, we introduce the notation used in the paper and also formally state the problem being studied. Section III focuses on the case where we have one degree of freedom and shows that in that case the DM tradeoff is successively refinable. In Section IV we characterize the rate tuples achievable for the parallel channel case. Section V gives some examples of finite block length codes that achieve the outage performance.

Preliminary results on the single degree of freedom case have appeared in [4]; the present paper introduces the two-stream outage formulation and extends the analysis to the parallel channel case.

II. Problem Statement

Our focus in this paper is on the quasi-static flat-fading channel where we transmit information coded over M_t transmit antennas and have M_r antennas at the receiver:

$$\mathbf{y}(k) = \mathbf{H}\mathbf{x}(k) + \mathbf{z}(k), \tag{1}$$

where $\mathbf{H}$ is M_r by M_t and has i.i.d. Rayleigh distributed entries, $\{\mathbf{x}(k)\}$ is the input sequence with transmit power constraint P, and $\{\mathbf{z}(k)\}$ is assumed to be additive white Gaussian noise with variance σ^2. Furthermore, the assumption is that the transmitter has no channel state information, whereas the receiver is able to perfectly track the channel (a common assumption, see for example [9]). This model is quite general:

- $M_t = M_r = 1$: SISO channel;
- $M_t = 1, M_r > 1$: SIMO channel;
- $M_t > 1, M_t = 1$: MISO channel;
- $M_t = M_r$, diagonal $\mathbf{H}$: parallel channel

The diversity-multiplexing tradeoff $d^*(r)$ is the high SNR version of the outage probability curve $p(R)$, scaled as follows:

$$\lim_{\mathsf{SNR}\to\infty} \frac{\log p(r \log \mathsf{SNR}))}{\log \mathsf{SNR}} = -d^*(r).$$

We use the notation:

$$p(r \log \mathsf{SNR}) \doteq \mathsf{SNR}^{-d^*(r)}$$

to denote this.

In [11], the following result on the diversity-multiplexing tradeoff $d^*(r)$ was established.

Theorem 2.1: (Zheng and Tse 2002) Let $K = \min(M_t, M_r)$. The tradeoff curve $d^*(r)$ is given by the piece-wise linear function connecting points in $(k, d^*(k)), k = 0, \ldots, K$ where

$$d^*(k) = (M_r - k)(M_t - k). \tag{2}$$

Now consider the scenario when we have two information streams with different reliability levels

(Figure 2). We can define (r_H, d_L, r_L, d_L) as an achievable rate-diversity tuple if there exists a sequence $\{(R_H(\mathsf{SNR}), p_H(\mathsf{SNR}), r_L(\mathsf{SNR}), p_L(\mathsf{SNR}))\}$ such that $(R_H(\mathsf{SNR}), p_H(\mathsf{SNR}), r_L(\mathsf{SNR}), p_L(\mathsf{SNR}))$ is an achievable outage performance at each finite SNR and

$$R_H = r_H \log \mathsf{SNR}, \quad p_H \doteq \mathsf{SNR}^{-d_H},$$

$$R_L = r_L \log \mathsf{SNR}, \quad p_L \doteq \mathsf{SNR}^{-d_L}.$$

Note that if we examine the joint message $m = (m_H, m_L)$, the total multiplexing rate is $r = r_H + r_L$ and the diversity is $d = \min(d_H, d_L) = d_L$. Therefore, in a baseline comparison with a single-stream code, the rate-diversity operating point is $(r_H + r_L, d_L)$. If $(r, d^*(r))$ is the optimal single-layer rate-diversity point predicted by Theorem 2.1, then $d_L \leq d^*(r_H + r_L)$. Of course, we would like $d_H > d^*(r_H + r_L)$, otherwise there would be no point in splitting the data into two streams in the first place. This means that a rate $r_H + r_L$ code with diversity d_L has a high-diversity message set embedded within it. The problem explored in this paper is the characterization of the achievable rate-diversity tuples (r_H, d_H, r_L, d_L).

Time sharing:: A simple strategy is to use a code that achieves points on the trade-off given in Theorem 2.1 and do time-sharing between those codes. Suppose we allocate a fraction α of the time to the high-priority stream and the rest to the low-priority stream. Then, the effective multiplexing rate for the high-priority stream is $\frac{r_H}{\alpha}$ and for the low-priority stream it is $\frac{r_L}{1-\alpha}$. Hence, we can achieve the tuple $r_H, d_H, r_L, d_L)$ where,

$$d_H = d^*(\frac{r_H}{\alpha}), \;\; d_L = d^*(\frac{r_L}{1-\alpha}), \tag{3}$$

where $d^*(r)$ is specified in Theorem 2.1. If we specialize this to the scalar case, *i.e.,* $M_t = 1 = M_r$, then we get the following tradeoff,

$$\frac{r_H}{1-d_H} + \frac{r_L}{1-d_L} = 1. \tag{4}$$

The question is whether and by how much we can improve upon time-sharing.

III. One degree of freedom: successive refinability

In this section we focus on the case when there is one degree of freedom, *i.e.,* $\min(M_t, M_r) = 1$. We consider superposition coding and define a power allocation across the two streams. We generate a random i.i.d. Gaussian codebook for each stream and allocate powers P_H, P_L respectively. Given the total power constraint P, the power allocation needs to satisfy $P_H + P_L = P$. In this section we analyze a successive decoding strategy, where the high-priority message m_H is decoded considering the codeword for the low-priority message m_L as part of the noise. Then the decoded codeword for the high-priority stream is subtracted from the received signal, and then the low-priority message m_L is decoded. Note that such a strategy just establishes an achievability result for the rate tuple (r_H, d_H, r_L, d_L).

$$\begin{aligned} \mathcal{O}_H(\mathsf{SNR}) &= \left\{\mathbf{h} : \log\left(1 + \frac{||\mathbf{h}||^2\mathsf{SNR_H}}{1+||\mathbf{h}||^2\mathsf{SNR_L}}\right) < R_H\right\} \quad (5) \\ \tilde{\mathcal{O}}_L(\mathsf{SNR}) &= \left\{\mathbf{h} : \log\left(1+||\mathbf{h}||^2\mathsf{SNR_L}\right) < R_L\right\} \\ p_H(SNR) &= \mathcal{P}\left\{\mathcal{O}_H(\mathsf{SNR})\right\}, \, p_L(\mathsf{SNR}) = \mathcal{P}\left\{\mathcal{O}_L(\mathsf{SNR})\right\}, \end{aligned}$$

where $\mathsf{SNR_H} = \frac{P_H}{\sigma^2}$, $\mathsf{SNR_L} = \frac{P_L}{\sigma^2}$, $\mathcal{O}_L(\mathsf{SNR}) = \mathcal{O}_H(\mathsf{SNR}) \bigcup \tilde{\mathcal{O}}_L(\mathsf{SNR})$ and $||\mathbf{a}||^2$ denotes the squared l_2 norm of $\mathbf{a}$. We design the power allocation to the streams such that $\frac{P_H}{P_L} = (P/\sigma^2)^\beta = \mathsf{SNR}^\beta$, $\beta \in [0,1]$. Then, at high SNR, we have $\mathsf{SNR_H} \doteq \mathsf{SNR}$, $\mathsf{SNR_L} \doteq \mathsf{SNR}^{1-\beta}$. We want to characterize the behavior of $p_H(\mathsf{SNR}), p_L(\mathsf{SNR})$ in the high SNR regime. Using this scheme, we can establish the following result.

Theorem 3.1: When $\min(M_t, M_r) = 1$, then the diversity-multiplexing tradeoff curve is successively refinable, i.e., for any multiplexing gains r_H and r_L such that $r_H + r_L \leq 1$, the diversity orders

$$\begin{aligned} d_H &= d^*(r_H), \\ d_L &= d^*(r_H + r_L) \end{aligned} \tag{6}$$

are achievable, where $d^*(r)$ is the optimal diversity order given in Theorem 2.1. Hence, the best possible performance can be achieved.

Proof: For the high-priority information stream, an outage occurs when,

$$\log\left(1 + \frac{||\mathbf{h}||^2\mathsf{SNR}}{1+||\mathbf{h}||^2\mathsf{SNR}^{1-\beta}}\right) < R_H = r_H \log(\mathsf{SNR}). \tag{7}$$

Hence if we choose $\beta > r_H$, then it can be shown that the outage probability is,

$$p_H(SNR) \doteq \mathcal{P}\left\{\mathbf{h} : \log\left(1+||\mathbf{h}||^2\mathsf{SNR}\right) < r_H \log(\mathsf{SNR})\right\}, \tag{8}$$

which implies that $d_H = d^*(r_H)$ is achievable. Similarly $\tilde{\mathcal{O}}$ as defined in (5) is

$$\left\{\mathbf{h} : \log\left(1+||\mathbf{h}||^2\mathsf{SNR}^{1-\beta}\right) < R_L = r_L \log(\mathsf{SNR})\right\}, \tag{9}$$

which shows that $d_L = d^*(\beta + r_L)$ is achievable since we can easily see that $p_L(\mathsf{SNR}) = \mathcal{P}\left\{\mathcal{O}_L(\mathsf{SNR})\right\} \doteq \mathcal{P}\left\{\tilde{\mathcal{O}}_L(\mathsf{SNR})\right\}$. Since $d^*(r)$ is an decreasing function of r and β can be chosen arbitrarily close to r_H, we see that any $d_L < d^*(r_H + r_L)$ is achievable. ■

A natural question to ask is whether the tradeoff is infinitely refinable, *i.e.,* if we allow for $Q > 2$ information streams would we still get optimality for each successive rate-sum? The following result answers this question in the positive.

Theorem 3.2: Let $\min(M_t, M_r) = 1$, then given Q information streams with multiplexing rates $r_1, \ldots, r_Q$, as long as $\sum_k r_k \leq 1$, the optimal diversity orders are achievable:

$$d_k = d^*(\sum_{i=1}^{k} r_i), \;\; k = 1, \ldots Q. \tag{10}$$

Therefore, we get infinite successive refinability when $\min(M_t, M_r) = 1$ and this means that we can design a variable rate coding scheme that achieves the multiplexing rate a particular channel realization can support and hence gets the same diversity order as the scheme which was pre-designed for that rate. Therefore we can design an "ideal" opportunistic code for this case.

The next natural question to ask is whether the successive refinability property holds for channels with more than one degree of freedom. In particular, this tries to answer the question whether this property has anything to do with the fact that for $\min(M_t, M_r) = 1$, the class of channels we encounter constitute degraded broadcast channels. This is the focus of Section IV.

IV. Parallel channels

In this section we study transmission over parallel channels, which are the simplest instantiation of channels with multiple degrees of freedom. This is described by (1) with $\mathbf{H}$ diagonal, with $M_t = M_r = K$, the number of sub-channels. The kth sub-channel faces a Rayleigh distributed fading gain of h_k and the fading gains are assumed to be independent across the K sub-channels. For this case, the diversity-multiplexing tradeoff achieved in a single stream setting is $d^*(r) = K - r$ [11]. What is the performance in a diversity-embedding setting?

Suppose we continue with the superposition coding strategy used for channels with one degree of freedom, using i.i.d. Gaussian codes for each stream, allocating a power of P_H, P_L respectively to the high and low-priority streams respectively. Moreover, we split the power equally across the sub-channels for both streams. In analogous to (5), for given target rates R_H and R_L for the two streams, the outage probabilities are given by:

$$\begin{aligned}
\mathcal{O}_H(\mathsf{SNR}) &= \left\{\mathbf{h} : \sum_{k=1}^{K} \log\left(1 + \frac{|h_k|^2\mathsf{SNR_H}}{1 + |h_k|^2\mathsf{SNR_L}}\right) < R_H\right\} \\
\tilde{\mathcal{O}}_L(\mathsf{SNR}) &= \left\{\mathbf{h} : \sum_{k=1}^{K} \log\left(1 + |h_k|^2\mathsf{SNR_L}\right) < R_L\right\}, \quad (11) \\
p_H(SNR) &= \mathcal{P}\left\{\mathcal{O}_H(\mathsf{SNR})\right\}, \, p_L(\mathsf{SNR}) = \mathcal{P}\left\{\mathcal{O}_L(\mathsf{SNR})\right\},
\end{aligned}$$

where $\mathcal{O}_L(\mathsf{SNR}) = \mathcal{O}_H(\mathsf{SNR}) \bigcup \tilde{\mathcal{O}}_L(\mathsf{SNR})$, $\mathsf{SNR_H} = \frac{P_H}{K\sigma^2}$, $\mathsf{SNR_L} = \frac{P_L}{K\sigma^2}$. We design the power allocation to the streams such that $\frac{P_H}{P_L} = (P/\sigma^2)^\beta = \mathsf{SNR}^\beta, \ \beta \in [0, 1]$. Then, at high SNR, we have $\mathsf{SNR_H} \doteq \mathsf{SNR}$, $\mathsf{SNR_L} \doteq \mathsf{SNR}^{1-\beta}$. By varying over all values of β and scaling the rates and the outage probabilities as a function of SNR as in the proof of Theorem 3.1, one can characterize the achievable tuples (r_H, d_H, r_L, d_L) of multiplexing and diversity gains. However, the full expression is not very insightful. To answer the question of whether successively refinability can be achieved, we consider the special case of $K = 2$ sub-channels and focus on one extreme point of the region.

Consider for a given multiplexing gain $r_H < 1$, that we require the high priority stream to have the diversity $d_H = d^*(r_H) = 2 - r_H$, i.e. as though it were on its own. We also are given $r_L < 1 - r_H$ and we want to characterize the maximal diversity order d_L that can be achieved. In order for the high-priority stream to have the maximum possible diversity, it can be shown from the first outage condition in (11) that β must be no smaller than r_H. Using this in the second outage condition, it can be derived that the diversity order is at most

$$d_L = 2 - 2r_H - r_L,$$

which is strictly less than the diversity $2 - r_H - r_L$ that can be achieved if a code was designed specifically for the joint stream (of multiplexing gain $r_H + r_L$. (Choosing β arbitrarily close to r_H can achieve this diversity.) Hence, successive refinability is not achieved.

But so far we have considered only a *specific* class of strategies, that of superposition coding using i.i.d. Gaussian codes for each stream and equal power allocation across the sub-channels. Can better performance be achieved by using other strategies? We examine this question by making a connection between our problem and the broadcast channel with degraded message sets.

Consider our problem of sending two messages (m_H, m_L) to a class of channels in $\mathcal{G} \subset \bar{\mathcal{O}}_H$ and message m_H to the set of channels $\bar{\mathcal{O}}_H$. We can show that this is equivalent to a broadcast problem with degraded message sets, with users in $\bar{\mathcal{O}}_H$. The broadcast problem is to deliver a common message m_H to *all* the users, but send a private message m_L to users in the subset $\mathcal{G} \subset \bar{\mathcal{O}}_H$. For the two user case, the capacity region of the parallel broadcast channel has been characterized in [5], where it was showed that superposition coding using i.i.d. Gaussian codes and optimal power allocation is indeed optimal. In our case we have many more than two users. However, the result suggests that a Gaussian superposition coding strategy might be optimal for our degraded message set problem as well. If so, at high SNR, it can be shown that the optimal power allocation is to put equal power across the sub-channels for each stream, which implies that the strategy outlined before may be optimal.

Even though one may not achieve successive refinability on the parallel channel, the diversity-embedded coding system should still be considered as strictly better than the single-stream system. Consider the rate-diversity tuple $(r_H, 2 - r_H, r_L, 2 - 2r_H - r_L)$, achievable by the strategy above. A single-stream system with multiplexing gain r_H achieves a diversity of $2 - r_H$, even with an optimal code. A diversity-embedded system can provide that performance (in the high-priority stream) but can in addition pump extra bits through when the channel is good. This means that there is slack in the original single-stream system and the diversity-embedded system intelligently exploits that slack.

V. Finite Block Length Codes

For the case where $M_t = 1$, we can devise a simple uncoded QAM scheme as illustrated in Figure 3 to attain the optimal performance as specified in Theorem 3.1. In order to see this notice that if we first place $2^{R_H} = \mathsf{SNR}^{r_H}$ QAM points and superpose on them $2^{R_L} = \mathsf{SNR}^{r_L}$ QAM points we get the clusters illustrated in Figure 3. For QAM constellations of size 2^R with power P, the minimum distance between signaling points is $d^2_{\text{min}} \approx \frac{P}{2^R}$. Hence if we choose $\mathsf{SNR}_H \doteq \mathsf{SNR}, \mathsf{SNR}_L \doteq \mathsf{SNR}^{1-\beta}$ with $\beta > r_H$, then we can easily see that the worst case minimum distance between the clusters, $d^{'}_{\text{min}}$ behaves like $d_{\text{min}}(H)$. Therefore, the information stream H it attains the same performance as if we sent uncoded QAM without the second stream L. Now, for $M_t = 1$, uncoded QAM attains optimal rate-diversity trade-off [10]. Therefore, when $M_t = 1$, for an arbitrary number of receive antennas M_r, the successive refinability is achievable for finite block length.

In the case when $M_t = 2, M_r = 1$, we know that the Alamouti scheme [1] achieves the entire rate-diversity trade-off curve [11]. In our case, we can use a superposed Alamouti scheme with uncoded QAM transmission to achieve successive refinability result of Theorem 3.1. The Alamouti code converts the two-transmit one receive antenna channel into the following pair of statistically identical scalar channels,

$$y_1 = (|h_1|^2 + |h_2|^2)^{1/2} x_1 + z_1, \quad y_2 = (|h_1|^2 + |h_2|^2)^{1/2} x_2 + z_2. \tag{12}$$

We can therefore send uncoded QAM with SNR^{r_H} points in stream H and SNR^{r_L} points of stream L superposed over it in each of the scalar channels. Using the same argument as above, if $\mathsf{SNR_L} \doteq \mathsf{SNR}^{1-\beta}$, and $\beta > r_H$, again the stream H attains the performance as if stream L was absent, and stream L can achieve diversity order $d^*(r_H + r_L)$. Hence it seems like the Alamouti scheme is canonical in this case as well.

For general MISO channels with $M_t > 2$, we can show a sufficient condition for finite length codes to achieve the outage bound given in Theorem 3.1. Consider a space-time code with $\mathbf{X} = \mathbf{X}_H + \frac{1}{\mathsf{SNR}^\beta}\mathbf{X}_L$, with $\mathbf{X}_H, \mathbf{X}_L \in \mathbb{C}^{M_t \times T}$, being individual space-time codes with multiplexing rates r_H, r_L respectively that satisfy

$$\lambda^2_{\min,H} \dot{>} \mathsf{SNR}^{1-r_H}, \quad \lambda^2_{\min,L} \dot{>} \mathsf{SNR}^{1-r_L}, \tag{13}$$

where $\lambda_{\min,H}, \lambda_{\min,L}$ are the smallest singular values of the codeword difference matrices $(\mathbf{X}_H - \mathbf{X}^{'}_H)$ and $(\mathbf{X}_L - \mathbf{X}^{'}_L)$ respectively. Then, with $\beta > r_H$, we can show that the average error probabilities behave as $P_{e,H}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d^*(r_H)}, P_{e,L}(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d^*(r_H + r_L)}$ matching the outage result in Theorem 3.1. Therefore, (13) gives the aforementioned sufficient conditions. In the context of single-layer space-time codes, has been recently established that for $T = M_t$, there exist space-time codes that satisfy the non-vanishing determinant (NVD) criterion required [6]. The NVD criterion is equivalent the condition in (13) (see pp 414 in [10]). Therefore, such algebraic codes can be used to also construct delay optimal diversity embedded codes using the above strategy.

VI. Discussion

In this paper we studied the characterization of rate tuples achievable when we desire multiple levels of reliability in a single user channel. When we have one degree of freedom *i.e.*, $\min(M_t, M_r) = 1$, we found that the rate-diversity trade-off is successively refinable. Even though for parallel channels this property may no longer hold, we still dominate a single stream coding. There are several open questions still remaining.

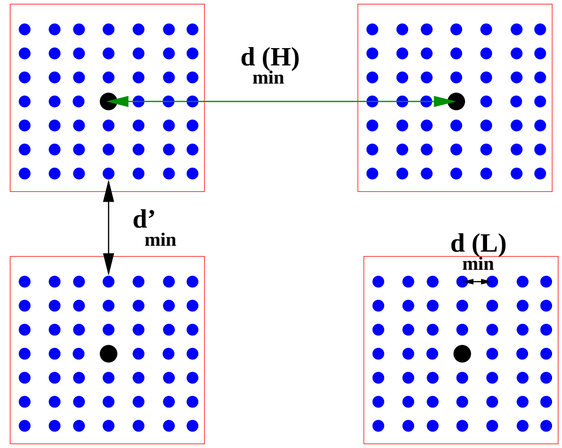


Fig. 3. Uncoded QAM transmission for $M_t = 1$ giving successive refinement of diversity.

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