Approximate capacity of Fading Gaussian Interference Channels with Point-to-Point codes

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Abstract

In this paper, we study the 2-user Gaussian interference-channel with feedback and fading links. We first show that for a class of fading models when no channel state information at transmitter (CSIT) is available, the rate-splitting schemes for static interference channel, when extended to the fading case, yield an approximate capacity region characterized to within a constant gap. We also show a constant-gap capacity result for the case without feedback. As rate-splitting requires superposition coding techniques, we next explore simpler schemes for the interference channel (with feedback) that do not use superposition, rate-splitting or joint-decoding. We demonstrate that point-to-point codes designed for inter-symbol-interference channels, along with time-sharing can be used to approximately achieve the entire rate region of the fading interference channel with symmetric fading statistics and feedback. We also characterize the gaps associated with some common fading models.

I. INTRODUCTION

The 2-user Gaussian interference channel (IC) is a simple model that captures the effect of interference in wireless networks. Significant progress has been made in the last decade in understanding the capacity of the static Gaussian IC. But in practice the links in the channel could be varying rather than static. In this paper, we study the 2-user Gaussian IC with fading links.

Previous works have characterized the capacity region to within a constant gap for the static Gaussian IC with and without feedback. The capacity of the 2-user Gaussian IC without feedback was characterized to within 1 bit in [3]. In [10], Suh *et al.* characterized the capacity of the Gaussian IC with feedback to within 2 bits. These results were based on the Han-Kobayashi

Shorter versions of this work appeared in [9], [8] with outline of proofs. This version has complete proofs. This work was supported in part by NSF grants 1514531 and 1314937.

scheme, where the transmitters split their messages into common and private parts and uses joint decoding at receivers.

The problem of characterizing capacity region for the case of the continuously fading channel with no channel state information at transmitter (CSIT) has not received much attention. The general Han-Kobayashi scheme for IC [2] can be applied, but it is complex to analyze due to the time-sharing involved. In [13], Wang *et al.* considered the bursty IC where the interference is either present or not present. In [12] Vahid *et al.* studied the binary fading model for the two-user IC, where the channel gains, the transmit signals and the received signals are in the binary field. In [5] Gou *et al.* proposed an interference neutralization scheme and showed a 2 degrees of freedom result for $2 \times 2 \times 2$ fading IC with full channel state information at the relays and destinations. In [6], Kang *et al.* considered interference alignment for the fading K-user IC with delayed feedback and showed a result of $\frac{2K}{K+2}$ degrees of freedom. Tuninetti [11] studied power allocation policies for fading Gaussian IC with CSIT and numerically showed that for Rayleigh fading, their scheme is close to optimal for some system parameters. In [4] Farsani showed that for fading Gaussian IC if each transmitter has knowledge of the instantaneous interference to noise ratio (inr) to the non-corresponding[1]receiver, the capacity region can be achieved within one bit.

In this paper, we first show that the Han-Kobayashi type rate-splitting schemes [2], [3], [10] can be extended to a class of fading models that satisfy a condition on the distribution of crosslink strengths. The condition effectively requires that a 'Jensens Gap' given by $\log\left(\mathbb{E}\left[inr\right]\right) - \mathbb{E}\left[\log\left(inr\right)\right]$ is uniformly bounded for the fading model. In particular, we will show that common fading models, including Rayleigh and Nakagami fading, satisfy the required condition. For such fading models we show that rate-splitting based on $\frac{1}{inr}$ for the static case in [3], [10] can be extended to schemes with rate-splitting based on $\frac{1}{\mathbb{E}[inr]}$ for the fading case to approximately obtain the whole capacity region. Our schemes use fixed power allocation to achieve any rate point, and for the feedback case (other than choosing the common and private message rates) we need to vary only a single power allocation parameter to achieve the whole inner bound, inheriting these properties from [3], [10].

For the feedback IC with symmetric fading statistics, we then devise a strategy that does not make use of rate splitting, superposition coding or joint decoding . Our scheme only uses point-

[1]For Tx1 the non-corresponding receiver is Rx2 and similarly for Tx2 the non-corresponding receiver is Rx1

to-point codes, and a feedback scheme based on amplify-and-forward relaying, similar to the one proposed in [10]. Through amplify-and-forward relaying of the feedback signal, the scheme effectively induces a 2-tap inter-symbol-interference (ISI) channel for one of the users and a point-to-point feedback channel for the other user. The work in [10] had similarly shown that an amplify-and-forward based feedback scheme can achieve the symmetric rate point, without using rate splitting. Our scheme can be considered as an extension to this scheme, which enables us to approximately achieve the **entire** capacity region of the symmetric IC with feedback.

In summary our main contributions are:

- Constant gap characterization of the capacity region for a large class of fading IC. In order to demonstrate this, we develop a *Jensen's Gap* characterization of fading models for approximate optimality with rate splitting schemes.
- Point-to-point codes for approximately achieving capacity region for IC with symmetric fading statistics and feedback.

The paper is organized as follows. In section II we describe the system setup and the notations. In section III we outline the main results on the approximate capacity region of fading ICs and our new scheme for symmetric fading IC which can be implemented using point to point codes. In section IV we provide the analysis for the approximate capacity results of fading ICs. In section V we provide the analysis for our new scheme for symmetric fading IC. In section VI we show that common fading models including Rayleigh and Nakagami fading satisfy the condition required for the approximate optimality of our schemes.

II. Model and Notation

We consider the two-user Gaussian fading IC

$$Y_1(t) = g_{11}(t)X_1(t) + g_{21}(t)X_2(t) + Z_1(t)$$

$$Y_2(t) = g_{12}(t)X_1(t) + g_{22}(t)X_2(t) + Z_2(t)$$

where $Y_i(t)$ is the channel output of receiver i (Rxi) at time t, where $X_i(t)$ is the input of transmitter i (Txi) at time t, $Z_i(t) \sim \mathcal{CN}(0,1)$ is complex AWGN noise process at Rxi, and $g_{ij}(t)$ is the time-variant random channel gain, i.i.d. across time, for $(i,j) \in \{1,2\}^2$. The channel gain processes $\{g_{ij}(t)\}$ are independent across links (i,j). At time t, the transmitters are assumed to have no knowledge of the channel gain realizations $\{g_{ij}(\tau)\}_{\tau \geq t}$, $(i,j) \in \{1,2\}^2$, *i.e.*, no future or present channel state is known at the transmitters, but Txi knows the past realizations of its direct link $\{g_{ii}(\tau)\}_{\tau<t}$. We assume $|g_{ij}(t)|^2$ is distributed according to P_{ij}, $(i,j) \in \{1,2\}^2$, and $\mathcal{P} := \{P_{ij}\}_{(i,j)\in\{1,2\}^2}$. We assume average power constraint $\frac{1}{n}\sum_{t=1}^{n}|X_i(t)|^2 \leq 1, i = 1,2$, at the transmitters, and assume Txi has a message $W_i \in \left[2^{NR_i}\right]$, for a block length of N, intended for Rxi for $i = 1,2$, and W_1 and W_2 are independent. We denote $SNR_i := \mathbb{E}\left[|g_{ii}|^2\right]$ for $i = 1,2$, and $INR_i := \mathbb{E}\left[|g_{ij}|^2\right]$ for $i \neq j$. For the instantaneous interference channel gains we use $inr_i := |g_{ij}|^2$, $i \neq j$.

Under the feedback model, after each reception, each receiver reliably feeds back the received symbol and the channel states to its corresponding transmitter, *i.e.*, Txi receives the corresponding channel outputs from Rxi up to time $t-1$ at time t, Y_i^{t-1}, and thus $X_i(t)$ is allowed to be a function of $\left(W_i, Y_i^{t-1}\right)$.

We define symmetric fading IC to be a fading IC such that $P_d := P_{11} = P_{22}$ and $P_c := P_{12} = P_{21}$. We then denote $SNR := \mathbb{E}\left[|g_d|^2\right]$, and $INR := \mathbb{E}\left[|g_c|^2\right]$, assuming $g_d \sim P_d$ and $g_c \sim P_c$, for the symmetric case.

We use the vector notation $\underline{g}_1 = [g_{11}, g_{21}]$, $\underline{g}_2 = [g_{22}, g_{12}]$ and $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$. For schemes involving multiple blocks (phases) we use the notation $X_k^{(i)N}$, where k is the user index, i is the block (phase) index and N is the number of symbols per block. The notation $X_k^{(i)}(j)$ indicates the j^{th} symbol in the i^{th} block (phase) of k^{th} user. For a complex number z, we use $\text{Re}(z)$ to indicate its real part. The natural logarithm is denoted by $\ln()$ and the logarithm with base 2 is denoted by $\log()$. Also we define $\log^+(\cdot) := \max(\log(\cdot), 0)$.

III. Main results

We first show that a simple extension of the existing rate-splitting-based schemes for the Gaussian IC is approximately optimal for the fading case, under a wide class of channel distributions. Then we proceed to develop a scheme based on point-to-point codes for the symmetric fading Gaussian IC.

A. Rate splitting for Gaussian fading ICs

Our main results on the rate splitting schemes pertain to a class of channel distributions that satisfy a certain regularity condition. In particular, the class of distributions under which we can give an approximate optimality guarantee with rate splitting have uniformly bounded Jensen's gap in log-magnitude. The following condition makes this precise.

Condition 1. *Given* $\mathcal{P} := \{P_{ij}\}_{(i,j)\in\{1,2\}^2}$*, there exists* $c > 0$ *such that for all* $a \geq 0$ *and all* $P \in \mathcal{P}$*,*

$$\log\left(a + \mathbb{E}\left[W\right]\right) - \mathbb{E}\left[\log\left(a + W\right)\right] \leq c,$$

where W *is distributed according to* P*.*

In section VI we show that common fading models including Rayleigh and Nakagami fading satisfy this condition. We also show that the bursty interference model of [13] does not satisfy this condition, due to the existence of a point mass of at zero.

The following two theorems summarize our results on rate splitting for Gaussian fading ICs. The schemes for both cases are based on Han-Kobayashi-type rate splitting schemes, with the power allocation performed based on *expected* interference strength, instead of the instantaneous one. The schemes will be described in more detail later in this section.

For both cases we obtain capacity gaps in terms of the constant c from Condition 1, with the following definition.

Definition 2. A rate region $\mathcal{R}$ achieves a capacity gap of δ if for any $(R_1, R_2) \in \mathcal{C}$, $(R_1 - \delta, R_2 - \delta) \in \mathcal{R}$, where $\mathcal{C}$ is the capacity region of the channel.

Theorem 3. *For a feedback Gaussian fading IC with the set of channel state distributions* $\mathcal{P}$ *satisfying Condition 1, the rate region* $\mathcal{R}_{FB}$ *described by* $(1)-(6)$ *is achievable for* $0 \leq |\rho|^2 \leq 1$*,* $0 \leq \theta < 2\pi$ *with* $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1 - |\rho|^2\right)$*:*

$$R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\,|\rho|^2\,\mathit{Re}\left(e^{i\theta} g_{11} g_{21}^*\right) + 1\right)\right] - 1 \tag{1}$$

$$R_1 \leq \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}\,|g_{11}|^2 + \lambda_{p2}\,|g_{21}|^2\right)\right] - 2 \tag{2}$$

$$R_2 \leq \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2\,|\rho|^2\,\mathit{Re}\left(g_{22}^* g_{12} e^{i\theta}\right) + 1\right)\right] - 1 \tag{3}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1 + \left(1 - |\rho|^2\right)|g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}\,|g_{22}|^2 + \lambda_{p1}\,|g_{12}|^2\right)\right] - 2 \tag{4}$$

$$\begin{aligned} R_1 + R_2 \leq\ & \mathbb{E}\left[\log\left(|g_{22}|^2 + |g_{12}|^2 + 2\,|\rho|^2\,\mathit{Re}\left(g_{22}^* g_{12} e^{i\theta}\right) + 1\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p1}\,|g_{11}|^2 + \lambda_{p2}\,|g_{21}|^2\right)\right] - 2 \end{aligned} \tag{5}$$

$$\begin{aligned} R_1 + R_2 \leq\ & \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2\,|\rho|^2\,\mathit{Re}\left(e^{i\theta} g_{11} g_{21}^*\right) + 1\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p2}\,|g_{22}|^2 + \lambda_{p1}\,|g_{12}|^2\right)\right] - 2 \end{aligned} \tag{6}$$

and the region $\mathcal{R}_{FB}$ *has a capacity gap of at most* $c + 2$ *bits.*

Proof: See subsection IV-A ■

Theorem 4. *For a non-feedback Gaussian fading IC with the set of channel state distributions $\mathcal{P}$ satisfying Condition 1, the rate region $\mathcal{R}_{NFB}$ described by (7) − (13) is achievable with $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1\right)$:*

$$R_1 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 1 \tag{7}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 1 \tag{8}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 2 \tag{9}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 2 \tag{10}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + |g_{12}|^2\right)\right] - 2 \tag{11}$$

$$\begin{aligned} 2R_1 + R_2 \leq\ & \mathbb{E}\left[\log\left(1 + |g_{11}|^2 + |g_{21}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + |g_{12}|^2\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + \lambda_{p2}|g_{21}|^2\right)\right] - 3 \end{aligned} \tag{12}$$

$$\begin{aligned} R_1 + 2R_2 \leq\ & \mathbb{E}\left[\log\left(1 + |g_{22}|^2 + |g_{12}|^2\right)\right] + \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2 + |g_{21}|^2\right)\right] \\ & + \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{22}|^2 + \lambda_{p1}|g_{12}|^2\right)\right] - 3 \end{aligned} \tag{13}$$

and the region $\mathcal{R}_{NFB}$ has a capacity gap of at most $c + 1$ bits.

Proof: See subsection IV-B ■

It is useful to view Theorems 3 and 4 in the context of the existing results for the corresponding static ICs. It is known that for Gaussian ICs with and without feedback, one can approximately achieve the capacity region by performing superposition coding and allocating a power to the private symbols that is inversely proportional to the strength of the interference caused at the unintended receiver. Consequently, the received interference power is at the noise level, and the private symbols can be safely treated as noise, incurring only a constant rate penalty. At first sight, such a strategy seems impossible for the fading IC, where the transmitters do not have instantaneous channel information. What Theorems 3 and 4 reveal is that if the channel gain distributions satisfy Condition 1, it is sufficient to perform power allocation based on the inverse of average interference strength to approximately achieve the capacity region.

We compare the symmetric rate point achievable for the non-feedback symmetric Gaussian Fading IC in Figure 1. The SNR is varied after fixing $\frac{\log(INR)}{\log(SNR)} = .5$. The simulation yields a

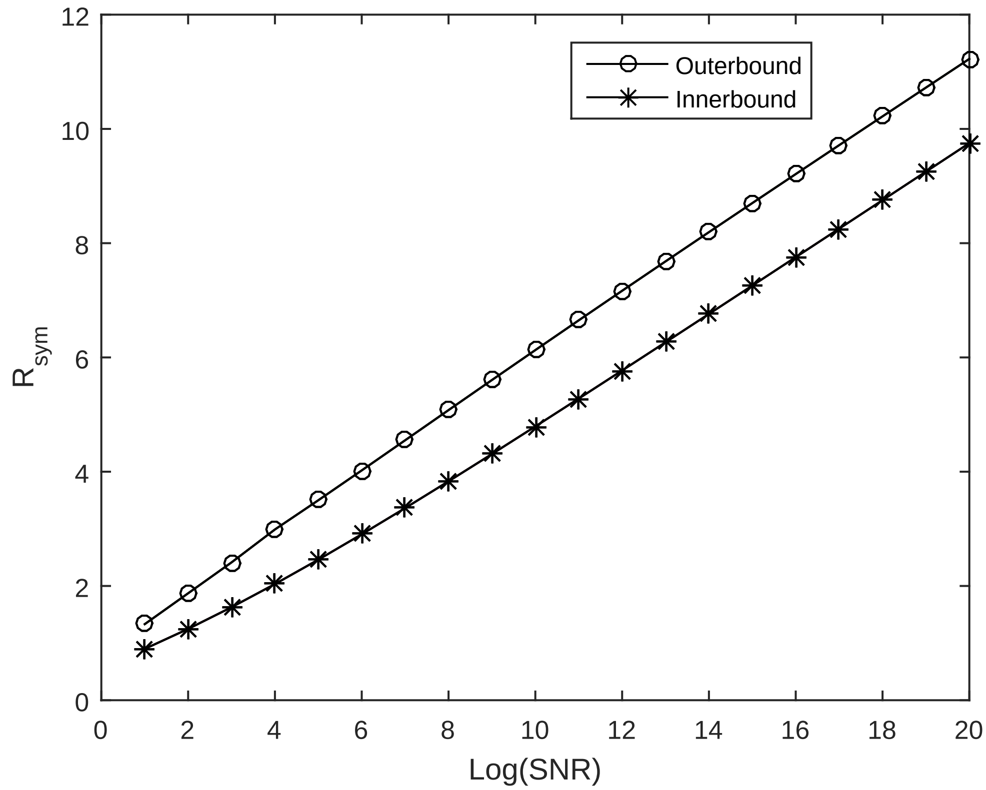

Figure 1. Comparison of outer and inner bounds with fixed $\frac{\log(INR)}{\log(SNR)} = .5$ for non-feedback symmetric Gaussian Fading IC at the symmetric rate point. The capacity gap is approximately 1.48 for high SNR from the numerics. Our theoretical analysis yields gap as 1.83 bits.

capacity gap of 1.48 bits compared to a capacity gap of $c+1 = 1.83$ bits which arises from our analysis for Gaussian Fading IC from Table VI-B in Section VI.

B. Description of the scheme with point-to-point codes

Although the rate splitting strategy of the previous section approximately achieves capacity for a wide class of channel distributions, it requires complex processing both at the transmitters and receivers, since it involves superposition coding at transmitters and joint decoding at the receivers. In the following subsection, we propose a strategy that does not make use of rate splitting, superposition coding or joint decoding for the feedback case, which achieves the entire capacity region for 2-user symmetric fading Gaussian interference channels to within a constant gap. Our scheme only uses point-to-point codes, and a feedback scheme based on amplify-and-forward relaying, similar to the one proposed in [10].

The main idea behind the scheme is to have one of the transmitters initially send a very densely modulated block of data, and then refine this information using feedback and amplify-and-forward relaying for the following blocks, in a fashion similar to the Schalkwijk-Kailath scheme [7], while treating the interference as noise. Such refinement effectively induces a 2-tap point-to-point inter-symbol-interference (ISI) channel at the unintended receiver, and a point-to-point feedback channel for the intended receiver. As a result, both receivers can decode their intended information using only point-to-point codes.

Consider the symmetric fading interference channel, where the channel statistics are symmetric and independent, *i.e.*, $g_{ii}(t) \sim g_d$, $g_{ij}(t) \sim g_c$, for $i \neq j$. We consider n transmission phases, each phase having a block length of N. Generate 2^{nNR_1} codewords $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$ i.i.d according to $\mathcal{CN}(0,1)$. Tx1 encodes its message $W_1 \in \left\{1, \ldots, 2^{nNR_1}\right\}$ onto $\left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$. For Tx2, generate 2^{nNR_2} codewords X_2^N and let it encode its message $W_2 \in \left\{1, \ldots, 2^{nNR_2}\right\}$ onto $X_2^{(1)N} = X_2^N$. Note that for Tx2, the coding block length for Tx2 is N, whereas it is nN for Tx1.

Tx1 sends $X_1^{(i)N}$ in phase i. Tx2 sends $X_2^{(1)N} = X_2^N$ in phase 1. At the beginning of phase $i > 1$, Tx2 receives

$$Y_2^{(i-1)N} = g_{22}^{(i-1)N} X_2^{(i-1)N} + g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}$$

from feedback. It can remove $g_d^{(i-1)N} X_2^{(i-1)N}$ from $Y_2^{(i-1)N}$ to obtain $g_c^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}$. Tx2 then transmits the resulting interference-plus-noise after power scaling

$$X_2^{(i)N} = \frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}}.$$

In phase $i > 1$ Rx2 receives

$$Y_2^{(i)N} = g_{22}^{(i)N} \left(\frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}} \right) + g_{12}^{(i)N} X_1^{(i)N} + Z_2^{(i)N}$$

and feeds it back to Tx2 for phase $i+1$. The transmission scheme is summarized in Table III-B. Note that for phase $i > 1$ Tx1 observes a block ISI channel since it receives

$$Y_1^{(i)N} = g_{11}^{(i)N} X_1^{(i)N} + g_{21}^{(i)N} \left(\frac{g_{12}^{(i-1)N} X_1^{(i-1)N} + Z_2^{(i-1)N}}{\sqrt{1+INR}} \right) + Z_1^{(i)N} \quad (14)$$

$$= g_{11}^{(i)N} X_1^{(i)N} + \left(\frac{g_{21}^{(i)N} g_{12}^{(i-1)N}}{\sqrt{1+INR}} \right) X_1^{(i-1)N} + \tilde{Z}_1^{(i)N} \quad (15)$$

where $\tilde{Z}_1^{(i)N} = Z_1^{(i)N} + \frac{g_{21}^{(i)N} Z_2^{(i-1)N}}{\sqrt{1+INR}}$.

Table I

TRANSMITTED SYMBOLS IN n-PHASE SCHEME FOR SYMMETRIC GAUSSIAN IC WITH FEEDBACK

User	Phase 1	Phase 2	.	.	Phase n
1	$X_1^{(1)N}$	$X_1^{(2)N}$	.	.	$X_1^{(n)N}$
2	$X_2^{(1)N}$	$\frac{g_{12}^{(1)N} X_1^{(1)N} + Z_2^{(1)N}}{\sqrt{1+INR}}$	.	.	$\frac{g_{12}^{(n-1)N} X_1^{(n-1)N} + Z_2^{(n-1)N}}{\sqrt{1+INR}}$

At the end of n blocks, Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots, Y_1^{(n)N}\right)$ and decodes by finding W_1 such that $\left(\mathbf{X_1}^N(W_1), \mathbf{Y_1}^N\right)$ is jointly typical, where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots, X_1^{(n)N}\right)$. At Rx2, channel outputs over n phases can be combined with appropriate scaling so that the interference-plus-noise at phases $\{1, \ldots, n-1\}$ are successively canceled, *i.e.*, an effective point-to-point channel can be generated through $\tilde{Y}_2^N = \sum_{i=1}^n \left(\prod_{j=i+1}^n \frac{-g_{22}^{(j)N}}{\sqrt{1+INR}} \right) Y_2^{(i)N}$ (see the analysis in the subsection V-B for details). Note that this can be viewed as a block version of the Schalkwijk-Kailath scheme [7] (and the references therein). Given the effective channel $\tilde{Y}_2^N$, the receiver can simply use point-to-point typicality decoding to recover W_2, treating the interference in phase n as noise.

Theorem 5. *If the set $\mathcal{P}$ of channel distributions of the fading IC satisfies Condition 1 with constant c, the rate pair*

$$(R_1, R_2) = \left(\log\left(1 + SNR + INR\right) - 2 - 3c, \mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1 + INR}\right]\right]\right)$$

is achievable by the scheme. The scheme together with switching the roles of users and time-sharing, achieves the capacity region of symmetric feedback IC within $2 + 3c$ bits.

Proof: See section V. ■

IV. ANALYSIS OF RATE SPLITTING SCHEMES

A. Proof of Theorem 3

Note that since the receivers know their respective incoming channel states, we can view the effective channel output at Rxi as the pair $\left(Y_i, \underline{g}_i\right)$. Then the block Markov scheme of [10] implies that the rate pairs (R_1, R_2) satisfying

$$R_1 \leq I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \tag{16}$$

$$R_1 \leq I\left(U_1; Y_2, \underline{g_2} | U, X_2\right) + I\left(X_1; Y_1, \underline{g_1} | U_1, U_2, U\right) \tag{17}$$

$$R_2 \leq I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \tag{18}$$

$$R_2 \leq I\left(U_2; Y_1, \underline{g_1} | U, X_1\right) + I\left(X_2; Y_2, \underline{g_2} | U_1, U_2, U\right) \tag{19}$$

$$R_1 + R_2 \leq I\left(X_1; Y_1, \underline{g_1} | U_1, U_2, U\right) + I\left(U, U_1, X_2; Y_2, \underline{g_2}\right) \tag{20}$$

$$R_1 + R_2 \leq I\left(X_2; Y_2, \underline{g_2} | U_1, U_2, U\right) + I\left(U, U_2, X_1; Y_1, \underline{g_1}\right) \tag{21}$$

for all $p(u)\, p(u_1|u)\, p(u_2|u)\, p(x_1|u_1, u)\, p(x_2|u_2, u)$ are achievable. We choose the input distribution according to

$$U \sim \mathcal{CN}\left(0, |\rho|^2\right), U_k \sim \mathcal{CN}\left(0, \lambda_{ck}\right), X_{pk} \sim \mathcal{CN}\left(0, \lambda_{pk}\right)$$

$$X_1 = e^{i\theta} U + U_1 + X_{p1}$$

$$X_2 = U + U_2 + X_{p2}$$

with $0 \leq |\rho|^2 \leq 1$, $0 \leq \theta < 2\pi$, $\lambda_{ck} + \lambda_{pk} = 1 - |\rho|^2$ and $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1 - |\rho|^2\right)$. Note that we have introduced an extra rotation θ for the first transmitter, which will become helpful in proving the capacity gap.

With this choice of λ_{pk} we perform the rate splitting according to the average inr in place of rate splitting based on the constant inr for static channels. On evaluating the terms in $(16)-(21)$ for this choice of input distribution, we get the inner bound described by $(1)-(6)$; the calculations are deferred to Appendix A.

An outer bound for the feedback case is given by $(22)-(27)$ with $0 \leq |\rho| \leq 1$:

$$R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2+|g_{21}|^2+2\text{Re}\left(\rho g_{11}g_{21}^*\right)+1\right)\right] \tag{22}$$

$$R_1 \leq \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)|g_{12}|^2\right)\right]+\mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)|g_{12}|^2}\right)\right] \tag{23}$$

$$R_2 \leq \mathbb{E}\left[\log\left(|g_{22}|^2+|g_{12}|^2+2\text{Re}\left(\rho g_{22}^*g_{12}\right)+1\right)\right] \tag{24}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)|g_{21}|^2\right)\right]+\mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{22}|^2}{1+\left(1-|\rho|^2\right)|g_{21}|^2}\right)\right] \tag{25}$$

$$\begin{aligned}R_1+R_2 \leq\ & \mathbb{E}\left[\log\left(|g_{22}|^2+|g_{12}|^2+2\text{Re}\left(\rho g_{22}^*g_{12}\right)+1\right)\right]\\ &+\mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)|g_{12}|^2}\right)\right]\end{aligned} \tag{26}$$

$$\begin{aligned}R_1+R_2 \leq\ & \mathbb{E}\left[\log\left(|g_{11}|^2+|g_{21}|^2+2\text{Re}\left(\rho g_{11}g_{21}^*\right)+1\right)\right]\\ &+\mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{22}|^2}{1+\left(1-|\rho|^2\right)|g_{21}|^2}\right)\right],\end{aligned} \tag{27}$$

The outer bounds can be easily derived following the proof techniques from [10] using $\mathbb{E}\left[X_1X_2^*\right]=\rho$, treating $\left(Y_i,\underline{g}_i\right)$ as output, and using the i.i.d property of the channels. The calculations are deferred to the Appendix B.

Claim 6. The gap between the inner bound $(1)-(6)$ and the outer bound $(22)-(27)$ for the feedback case is atmost $c+2$ bits.

Proof: The Condition 1 we imposed on the fading distribution and the rotation θ for the first transmitter become important in proving a constant gap capacity result. We compare the corresponding equations in outer and inner bounds. Denote the gap between the first outer bound and inner bound by δ_1, for the second pair denote the gap by δ_2, and so on. Choose θ in the inner bound to match $\arg(\rho)$ in the outer bound. We get

$$\begin{aligned}\delta_1 &= \mathbb{E}\left[\log\left(|g_{11}|^2+|g_{21}|^2+2|\rho|\,\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)+1\right)\right]\\ &\quad -\mathbb{E}\left[\log\left(|g_{11}|^2+|g_{21}|^2+2|\rho|^2\,\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)+1\right)\right]+1\\ &= \mathbb{E}\left[\log\left(\frac{1+|g_{11}|^2+|g_{21}|^2+2|\rho|\,\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{1+|g_{11}|^2+|g_{21}|^2+2|\rho|^2\,\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}\right)\right]+1\\ &= \mathbb{E}\left[\log\left(\frac{1+\frac{1}{|g_{11}|^2+|g_{21}|^2}+|\rho|\left(\frac{2\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{|g_{11}|^2+|g_{21}|^2}\right)}{1+\frac{1}{|g_{11}|^2+|g_{21}|^2}+|\rho|^2\left(\frac{2\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{|g_{11}|^2+|g_{21}|^2}\right)}\right)\right]+1.\end{aligned}$$

We have

$$\left|\frac{2\text{Re}\left(e^{i\theta}g_{11}g_{21}^*\right)}{|g_{11}|^2+|g_{21}|^2}\right| = \frac{\left|e^{-i\theta}g_{11}^*g_{21}+e^{i\theta}g_{11}g_{21}^*\right|}{|g_{11}|^2+|g_{21}|^2} \leq 1,$$

hence we call $\frac{e^{-i\theta}g_{11}^*g_{21}+e^{i\theta}g_{11}g_{21}^*}{|g_{11}|^2+|g_{21}|^2} = \sin\phi$ and let $|g_{11}|^2+|g_{21}|^2 = r^2$. Therefore

$$\delta_1 = \mathbb{E}\left[\log\left(\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi}\right)\right]+1.$$

If $\sin\phi < 0$, then

$$\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi} \leq 1.$$

If $\sin\phi > 0$, then

$$\frac{1+\frac{1}{r^2}+|\rho|\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi} = 1+\frac{\left(|\rho|-|\rho|^2\right)\sin\phi}{1+\frac{1}{r^2}+|\rho|^2\sin\phi} \leq 2$$

since $0 \leq \left(|\rho|-|\rho|^2\right)\sin\phi \leq 1$ and $1+\frac{1}{r^2}+|\rho|^2\sin\phi > 1$. Hence $\delta_1 \leq 2$. Now we consider the gap δ_2 between the second inequality (23) of the outer bound and the second inequality (2) of the inner bound.

$$\begin{aligned}
\delta_2 &= \mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)|g_{12}|^2}\right)\right] - \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]+2 \\
&\overset{(a)}{\leq} \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)INR_1+\left(1-|\rho|^2\right)|g_{11}|^2\right)\right] - \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)INR_1\right)\right]+c \\
&\quad - \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]+2 \\
&\leq \mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)INR_1}\right)\right] - \mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2\right)\right]+2+c
\end{aligned} \tag{28}$$

where (a) follows from Condition 1 on the distribution of $|g_{ij}|^2$ and Jensen's inequality. We have $\lambda_{p1} = \min\left(\frac{1}{INR_1}, 1-|\rho|^2\right)$; on considering the cases $\lambda_{p1} = 1-|\rho|^2$ and $\lambda_{p1} = \frac{1}{INR_1}$ separately, it can be shown that

$$1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)INR_1} < 1+\lambda_{p1}|g_{11}|^2. \tag{29}$$

Hence $\delta_2 \leq c+2$ follows. By inspection of the other bounding inequalities we get

$$\delta_3 \leq 2$$

$$\delta_4 \leq c+2$$

$$\delta_5 \leq 3+c$$

$$\delta_6 \leq 3+c.$$

Hence it follows that the capacity gap is at most $c+2$ bits. ■

B. Proof of Theorem 4

From [2] we obtain that a Han-Kobayashi scheme for IC can achieve the following rate region for all $p(u_1)\,p(u_2)\,p(x_1|u_1)\,p(x_2|u_2)$. Note that we use $\left(Y_i, \underline{g_i}\right)$ instead of (Y_i) in the actual result from [2] to account for the fading.

$$R_1 \leq I\left(X_1; Y_1, \underline{g_1}|U_2\right) \tag{30}$$

$$R_2 \leq I\left(X_2; Y_2, \underline{g_2}|U_1\right) \tag{31}$$

$$R_1 + R_2 \leq I\left(X_2, U_1; Y_2, \underline{g_2}\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2\right) \tag{32}$$

$$R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2\right) \tag{33}$$

$$R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}|U_1\right) + I\left(X_2, U_1; Y_2, \underline{g_2}|U_2\right) \tag{34}$$

$$2R_1 + R_2 \leq I\left(X_1, U_2; Y_1, \underline{g_1}\right) + I\left(X_1; Y_1, \underline{g_1}|U_1, U_2\right) + I\left(X_2, U_1; Y_2, \underline{g_2}|U_2\right) \tag{35}$$

$$R_1 + 2R_2 \leq I\left(X_2, U_1; Y_2, \underline{g_2}\right) + I\left(X_2; Y_2, \underline{g_2}|U_1, U_2\right) + I\left(X_1, U_2; Y_1, \underline{g_1}|U_1\right). \tag{36}$$

Now similar to that in [3], choose the Gaussian input distribution

$$U_k \sim \mathcal{CN}\left(0, \lambda_{ck}\right), \quad X_{pk} \sim \mathcal{CN}\left(0, \lambda_{pk}\right), \qquad k \in \{1, 2\}$$

$$X_1 = U_1 + X_{p1}$$

$$X_2 = U_2 + X_{p2}$$

where $\lambda_{ck} + \lambda_{pk} = 1$ and $\lambda_{pk} = \min\left(\frac{1}{INR_k}, 1\right)$. Here we introduced the rate splitting using the average inr. On evaluating the region described by $(30) - (36)$ with this choice of input distribution, we get the region described by $(7) - (13)$; the computations are similar to that of the feedback case.

An outer bound for the non-feedback case is given by $(37)-(43)$

$$R_1 \leq \mathbb{E}\left[\log\left(1+|g_{11}|^2\right)\right] \tag{37}$$

$$R_2 \leq \mathbb{E}\left[\log\left(1+|g_{22}|^2\right)\right] \tag{38}$$

$$R_1+R_2 \leq \mathbb{E}\left[\log\left(1+|g_{22}|^2+|g_{12}|^2\right)\right]+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right] \tag{39}$$

$$R_1+R_2 \leq \mathbb{E}\left[\log\left(1+|g_{11}|^2+|g_{21}|^2\right)\right]+\mathbb{E}\left[\log\left(1+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right] \tag{40}$$

$$R_1+R_2 \leq \mathbb{E}\left[\log\left(1+|g_{21}|^2+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]+\mathbb{E}\left[\log\left(1+|g_{12}|^2+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right] \tag{41}$$

$$\begin{aligned}2R_1+R_2 \leq\ & \mathbb{E}\left[\log\left(1+|g_{11}|^2+|g_{21}|^2\right)\right]+\mathbb{E}\left[\log\left(1+|g_{12}|^2+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right]\\ &+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]\end{aligned} \tag{42}$$

$$\begin{aligned}R_1+2R_2 \leq\ & \mathbb{E}\left[\log\left(1+|g_{22}|^2+|g_{12}|^2\right)\right]+\mathbb{E}\left[\log\left(1+|g_{21}|^2+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]\\ &+\mathbb{E}\left[\log\left(1+\frac{|g_{22}|^2}{1+|g_{21}|^2}\right)\right],\end{aligned} \tag{43}$$

The outer bounds easily follow from the results in [3] by modifying them for the fading case by treating $\left(Y_i, \underline{g}_i\right)$ as output, and using the i.i.d property of the channels, and by using the outer bounds (26) and (27) for feedback case after setting $\mathbb{E}\left[X_1X_2^*\right]=\rho=0$.

Claim 7. The gap between the inner bound $(7)-(13)$ and the outer bound $(37)-(43)$ for the feedback case is atmost $c+1$ bits.

Proof: The proof for the capacity gap again uses the Condition 1 on the fading distribution.Denote the gap between the first outer bound (37) and first inner bound (7) by δ_1, δ_2 for the second pair and so on. Clearly $\delta_1 \leq 1$ and $\delta_2 \leq 1$. Now

$$\begin{aligned}\delta_3 &= 2+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+|g_{12}|^2}\right)\right]-\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2+\lambda_{p2}|g_{21}|^2\right)\right]\\ &\overset{(a)}{\leq} 2+\mathbb{E}\left[\log\left(1+\frac{|g_{11}|^2}{1+INR_1}\right)\right]+c-\mathbb{E}\left[\log\left(1+\lambda_{p1}|g_{11}|^2\right)\right].\end{aligned}$$

The step (a) follows from Jensen's inequality and Condition 1 on $|g_{12}|^2$. We have $\lambda_{p1} = \min\left(\frac{1}{INR_1}, 1\right) \geq \frac{1}{INR_1+1}$, hence

$$\mathbb{E}\left[\log\left(1 + \frac{|g_{11}|^2}{1+INR_1}\right)\right] - \mathbb{E}\left[\log\left(1 + \lambda_{p1}|g_{11}|^2\right)\right] \leq 0.$$

Therefore $\delta_3 \leq 2 + c$. Similarly one can show

$$\delta_4 \leq 2 + c$$

$$\delta_5 \leq 2 + 2c$$

$$\delta_6 \leq 3 + 2c$$

$$\delta_7 \leq 3 + 2c.$$

For δ_5, δ_6, and δ_7 we have to use the Condition 1 twice and hence $2c$ appears. Now it follows that the capacity gap is at most $c + 1$ bits. ■

V. Analysis of Point-to-Point Codes for Symmetric Gaussian Fading ICs

We provide the analysis for the scheme described in subsection III-B going through the decoding at the two receivers and then at the capacity gap for the region achievable using the scheme we developed.

A. Decoding at Rx1

At the end of n blocks Rx1 collects $\mathbf{Y_1}^N = \left(Y_1^{(1)N}, \ldots Y_1^{(n)N}\right)$ and decodes W_1 such that $\left(\mathbf{X_1}^N(W_1), \mathbf{Y_1}^N\right)$ is jointly typical, where $\mathbf{X_1}^N = \left(X_1^{(1)N}, \ldots X_1^{(n)N}\right)$. Using standard techniques it follows that for the n-phase scheme as $N \to \infty$ user 1 can achieve the rate $\frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$ where $|K_{\mathbf{Y_1}}(n)|$ denotes the determinant of covariance matrix for the the n-phase scheme defined in the following pattern

$$K_{\mathbf{Y_1}}(1) = \left[1 + |g_{11}(1)|^2 + |g_{21}(1)|^2\right]$$

$$K_{\mathbf{Y_1}}(2) = \begin{bmatrix} |g_{11}(2)|^2 + \frac{|g_{21}(2)|^2\left(|g_{12}(1)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(1)g_{21}(2)g_{12}(1)}{\sqrt{1+INR}} \\ \frac{g_{11}(1)g_{21}^*(2)g_{12}^*(1)}{\sqrt{1+INR}} & |g_{11}(1)|^2 + |g_{21}(1)|^2 + 1 \end{bmatrix}$$

$$K_{\mathbf{Y_1}}(3) = \begin{bmatrix} |g_{11}(3)|^2 + \frac{|g_{21}(3)|^2\left(|g_{12}(2)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(2)g_{21}(3)g_{12}(2)}{\sqrt{1+INR}} & 0 \\ \frac{g_{11}(2)g_{21}^*(3)g_{12}^*(2)}{\sqrt{1+INR}} & |g_{11}(2)|^2 + \frac{|g_{21}(2)|^2\left(|g_{12}(1)|^2+1\right)}{1+INR} + 1 & \frac{g_{11}^*(1)g_{21}(2)g_{12}(1)}{\sqrt{1+INR}} \\ 0 & \frac{g_{11}(1)g_{21}^*(2)g_{12}^*(1)}{\sqrt{1+INR}} & |g_{11}(1)|^2 + |g_{21}(1)|^2 + 1 \end{bmatrix}$$

where $g_{11}(i) \sim g_d$ i.i.d and $g_{12}(i), g_{12}(i) \sim g_c$ i.i.d. Letting $n \to \infty$, Rx1 can achieve the rate $R_1 = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$. We need to evaluate $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log\left(\frac{|K_{\mathbf{Y_1}}(n)|}{|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)|}\right)\right]$. The following lemma gives an upper bound on $\frac{1}{n}\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right]$.

Lemma 8.

$$\frac{1}{n}\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right] \geq \frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) - 3c$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ is obtained from $K_{\mathbf{Y_1}}(n)$ by replacing $g_{12}(i)$'s,$g_{21}(i)$'s with $\sqrt{INR}$ and $g_{11}(i)$'s with $\sqrt{SNR}$.

Proof: The proof involves expanding the matrix determinant and repeated application of the Condition 1. The details are given in Appendix C. ■

Subsequently we use the following lemma in bounding $\lim_{n\to\infty} \frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right)$.

Lemma 9. *If* $A_1 = [|a|], A_2 = \begin{bmatrix} |a| & b \\ b^* & |a| \end{bmatrix}, A_3 = \begin{bmatrix} |a| & b & 0 \\ b^* & |a| & b \\ 0 & b^* & |a| \end{bmatrix}, A_4 = \begin{bmatrix} |a| & b & 0 & 0 \\ b^* & |a| & b & 0 \\ 0 & b^* & |a| & b \\ 0 & 0 & b^* & |a| \end{bmatrix}$

etc. with $|a|^2 > 4|b|^2$, then

$$\liminf_{n\to\infty} \frac{1}{n}\log\left(|A_n|\right) \geq \log\left(\frac{|a|}{2}\right).$$

Proof: The proof is given in Appendix D. ■

For the n-phase scheme, the $\left|\hat{K}_{\mathbf{Y_1}}(n)\right|$ matrix has the form A_n, as defined in Lemma 9 after identifying $|a| = 1 + INR + SNR$ and $b = \frac{\sqrt{SNR}INR}{\sqrt{1+INR}}$. Note that with this choice $|a|^2 > 4|b|^2$ holds due to AM-GM inequality. Hence we have

$$\liminf_{n\to\infty} \frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) \geq \log\left(\frac{1 + INR + SNR}{2}\right) \tag{44}$$

using Lemma 9.Also, $K_{\mathbf{Y_1}|\mathbf{X_1}}(n)$ is a diagonal matrix of the form

$$K_{\mathbf{Y_1}|\mathbf{X_1}}(n) = \text{diag}\left(\frac{|g_{21}(n)|^2}{1+INR} + 1, \frac{|g_{21}(n-1)|^2}{1+INR} + 1, \ldots, \frac{|g_{21}(2)|^2}{1+INR} + 1, |g_{21}(1)|^2 + 1\right)$$

Hence using Jensen's inequality

$$\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(\left|K_{\mathbf{Y_1}|\mathbf{X_1}}(n)\right|\right)\right] \leq \limsup_{n\to\infty}\frac{1}{n}\log\left(\left(\frac{INR}{1+INR}+1\right)^{n-1}(1+INR)\right) \tag{45}$$

$$= \log\left(\frac{INR}{1+INR}+1\right) \tag{46}$$

$$\leq 1 \tag{47}$$

Hence

$$R_1 \leq \log\left(1+INR+SNR\right) - 3c - 2 \tag{48}$$

is achievable.

B. Decoding at Rx2

For user 2 we can use a block variant of Schalkwijk-Kailath scheme [7] to achieve $R_2 = \mathbb{E}\left[\log^+\left(\frac{|g_d|^2}{1+INR}\right)\right]$. The key idea is that the interference-plus-noise sent in subsequent slots can indeed refine the symbols of the previous slot. The chain of refinement over n phases compensate for the fact that the information symbols are sent only in the first phase. We have

$$Y_2^{(1)N} = g_{22}^{(1)N}X_2^N + g_{12}^{(1)N}X_1^{(1)N} + Z_2^{(1)N} \tag{49}$$

and

$$Y_2^{(i)N} = g_{22}^{(i)N}\left(\frac{g_{12}^{(i-1)N}X_1^{(i-1)N}+Z_2^{(i-1)N}}{\sqrt{1+INR}}\right) + g_{12}^{(i)N}X_1^{(i)N} + Z_2^{(i)N} \tag{50}$$

for $i > 1$. Now let

$$\begin{aligned} Y_2^N &= \sum_{i=1}^{n}\left(\prod_{j=i+1}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)Y_2^{(i)N} \\ &= g_{22}^{(1)N}\left(\prod_{j=i+1}^{n}\frac{-g_{22}^{(j)N}}{\sqrt{1+INR}}\right)X_2^N + g_{12}^{(n)N}X_1^{(n)N} + Z_2^{(n)N}. \end{aligned} \tag{51}$$

Now Rx2 decodes for its message from Y_2^N. Hence Rx2 can achieve the rate

$$R_2 \leq \liminf_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(1+\left(\prod_{j=i+1}^{n}\frac{\left|g_{22}\left(j\right)\right|^2}{1+INR}\right)\frac{\left|g_{22}\left(1\right)\right|^2}{1+\left|g_{12}\left(n\right)\right|^2}\right)\right] \tag{52}$$

where $g_{22}\left(1\right),\ldots,g_{22}\left(n\right) \sim g_d$ being i.i.d and $g_{12}\left(n\right) \sim g_c$.Hence it follows that

$$R_2 \leq \mathbb{E}\left[\log^+\left(\frac{\left|g_d\right|^2}{1+INR}\right)\right] \tag{53}$$

is achievable.

C. Capacity gap

We can obtain the following outer bounds from Theorem 3 for the special case of symmetric fading statistics.

$$R_1, R_2 \leq \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \tag{54}$$

$$R_1 + R_2 \leq \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1 + |g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 2\,|g_d|\,|g_c| + 1\right)\right] \tag{55}$$

The outer bounds reduce to a pentagonal region with two non-trivial corner points (see Figure 2). Our n-phase scheme can achieve the two corner points within $2 + 3c$ bits for each user. The proof is using Condition 1 and is deferred to Appendix E.

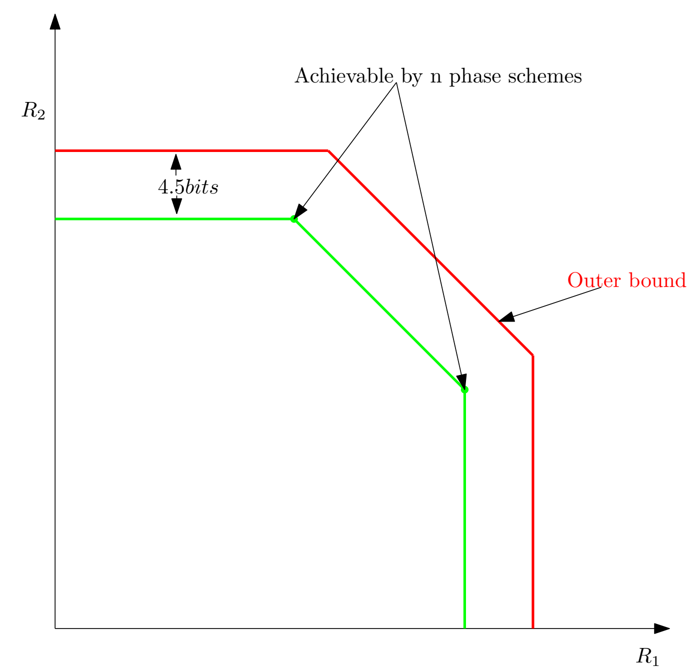

Figure 2. Illustration of bounds for capacity region for symmetric fading Gaussian IC. The corner points of the outer bound can be approximately achieved by our n-phase schemes. The gap is approximately 4.5 bits for the Rayleigh fading case.

Note that our analysis for R_1 can be easily modified to obtain a closed form approximate expression for the 2-tap fading ISI channel capacity, described by

$$Y(t) = g_d(t)\,X(t) + g_c(t)\,X(t-1) + Z(t)\,, \tag{56}$$

as a by-product. This gives rise to the following corollary on the capacity of fading ISI channels.

Corollary 10. *If there exists* c *that satisfies the Condition 1 for* g_d, g_c*, then the capacity* C_{F-ISI} *of the 2-tap fading ISI channel* (56) *is bounded by* $\log(1+SNR+INR)+1 \geq C_{F-ISI} \geq \log(1+SNR+INR)-1-3c$.

VI. Fading models

Here we discuss the fading models that satisfy the required Condition 1. The following lemma converts Condition 1 to a simpler form.

Lemma 11. *A set of channel distributions* $\mathcal{P}$ *satisfies Condition 1 with constant* c *if and only if for all* $P \in \mathcal{P}$,

$$\mathbb{E}\left[\log\left(W'\right)\right] \geq -c,$$

where $W' = \frac{W}{\mathbb{E}[W]}$*, and* W *is distributed according to* P.

Proof: We first note that $\phi(a) = \log(a+\mathbb{E}[W]) - \mathbb{E}[\log(a+W)] \geq 0$ due to Jensen's inequality. Taking derivative with respect to a and again using Jensen's inequality we get

$$(\ln 2)\,\phi'(a) = \frac{1}{a+\mathbb{E}[W]} - \mathbb{E}\left[\frac{1}{a+W}\right] \leq 0. \tag{57}$$

Hence $\phi(a)$ achieves the maximum value at $a=0$ in the range $[0,\infty)$. Hence we have the equivalent condition

$$\log(\mathbb{E}[W]) - \mathbb{E}[\log(W)] \leq c, \tag{58}$$

which is equivalent to

$$\mathbb{E}\left[\log\left(W'\right)\right] \geq -c. \tag{59}$$

■

Hence it follows that for any distribution that has a point mass at 0 (for example, bursty interference model [13]), we cannot guarantee a constant capacity gap, since it has $\mathbb{E}[\log(W')] = -\infty$. Now we discuss a few distributions that can be easily shown to satisfy the required condition for the scheme.

A. Gamma distribution

Gamma distribution generalizes some of the commonly used fading models, including Rayleigh and Nakagami fading. The probability density function for Gamma distribution is given by

$$f(w) = \frac{w^{k-1}e^{-\frac{w}{\theta}}}{\theta^k\Gamma(k)}$$

for $w > 0$, where $k > 0$ is the shape parameter, and $\theta > 0$ is the scale parameter.

Proposition 12. *If the elements of $\mathcal{P}$ are Gamma distributed with shape parameter k, $\mathcal{P}$ satisfies Condition 1 with constant $c = \frac{\log(e)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right)$ for any $0 < \alpha \leq k$.*

Proof: Using Lemma 11, it is sufficient to prove $\mathbb{E}\left[\log\left(W'\right)\right] \geq \frac{\log(e)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right)$, where W' is Gamma distributed with $k > 0$ and $\mathbb{E}\left[W\right]' = 1$. It is known for the Gamma distribution that $\mathbb{E}\left[W\right] = k\theta$ and $\mathbb{E}\left[\ln\left(W\right)\right] = \psi\left(k\right) + \ln\left(\theta\right)$, where ψ is the digamma function. Therefore

$$-\mathbb{E}\left[\log\left(W'\right)\right] = \log(e)\left(\ln\left(k\right) - \psi\left(k\right)\right) \tag{60}$$

We first use the following property of digamma function

$$\psi\left(k\right) = \psi\left(k+1\right) - \frac{1}{k}, \tag{61}$$

and then use the inequality from [1]

$$\ln\left(k + \frac{1}{2}\right) < \psi\left(k+1\right) < \ln\left(k + e^{-\gamma}\right). \tag{62}$$

Hence

$$\begin{aligned} -\mathbb{E}\left[\log\left(W'\right)\right] &< \log(e)\left(\ln\left(k\right) - \ln\left(k + \frac{1}{2}\right) + \frac{1}{k}\right) \\ &= \frac{\log\left(e\right)}{\alpha} - \log\left(1 + \frac{1}{2\alpha}\right). \end{aligned} \tag{63}$$

The last step follows because the function involved is decreasing in k in the range $(0, \infty)$ and since it is assumed $0 < \alpha \leq k$. ■

Corollary 13. *If the elements of $\mathcal{P}$ are Rayleigh distributed, $\mathcal{P}$ satisfies Condition 1 with constant $c = 0.86$.*

Proof: In Rayleigh fading model the $\left|g_{ij}\right|^2$ is exponentially distributed with mean INR_i. The exponential distribution itself is a special case of Gamma distribution with $k = 1$. Substituting $\alpha = 1$ in (63) we get $\mathbb{E}\left[\log\left(W'\right)\right] > -0.86$. ■

The constant for Nakagami fading can be obtained as a special case of the Gamma distribution; in this case the capacity gap will depend upon the parameters used in the model.

B. Weibull distribution

The probability density function for Weibull distribution is given by

$$f(w) = \frac{k}{\lambda}\left(\frac{w}{\lambda}\right)^{k-1} e^{-(w/\lambda)^k}$$

for $x > 0$ with $k, \lambda > 0$.

Proposition 14. *If the elements of $\mathcal{P}$ are Weibull distributed with parameter k, $\mathcal{P}$ satisfies Condition 1 with constant $c = \frac{\gamma \log(e)}{\alpha} + \log\left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right)$ for any $0 < \alpha \le k$, where γ is Euler's constant.*

Proof: For Weibull distributed W, we have $\mathbb{E}[W] = \lambda\Gamma\left(1 + \frac{1}{k}\right)$ and $\mathbb{E}[\ln(W)] = \ln(\lambda) - \frac{\gamma}{k}$, where $\Gamma(\cdot)$ denotes the gamma function and γ is the Euler's constant. Hence for $0 < \alpha \le k$, it follows that

$$-\mathbb{E}[\log(W')] \le \frac{\gamma \log(e)}{\alpha} + \log\left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right). \tag{64}$$

Using Lemma 11 concludes the proof. ■

Note that exponential distribution can be specialized from Weibull distribution as well, by setting $k = 1$. Hence we get the tighter gap in the following corollary.

Corollary 15. *If the elements of $\mathcal{P}$ are Rayleigh distributed, $\mathcal{P}$ satisfies Condition 1 with constant* $c = 0.83$.

In the following table we summarize the values of c in Condition 1 for different distributions.

Table II

VALUE OF c IN CONDITION 1 FOR DIFFERENT DISTRIBUTIONS

Fading Model	Value of c
Rayleigh	0.83
Gamma $k = 1$	0.86
Gamma $k = 2$	0.40
Gamma $k = 3$	0.26
Weibull $k = 1$	0.83
Weibull $k = 2$	0.24
Weibull $k = 3$	0.11

C. Other distributions

Here we give a lemma that can be used to verify whether a given fading model satisfies Condition 1.

Lemma 16. *If the cumulative distribution function $F(w)$ of W satisfies $F(w) \leq aw^b$ over $w \in [0, \epsilon]$ for some $a \geq 0$, $b > 0$, and $0 < \epsilon \leq 1$, then*

$$\mathbb{E}\left[\ln\left(W\right)\right] \geq \ln\left(\epsilon\right) + a\epsilon^b \ln\left(\epsilon\right) - \frac{a\epsilon^b}{b}. \tag{65}$$

Proof: The condition in this lemma ensures that the probability density function $f(w)$ grows slow enough as $w \to 0^-$ so that $f(w)\ln(w)$ is integrable at 0. Also the behavior for large values of w is not relevant here, since we are looking for a lower bound on $\mathbb{E}\left[\ln\left(W\right)\right]$. The detailed proof is given in Appendix F. ■

Hence if the cumulative distribution of the channel gains grow polynomially in a neighborhood of 0, the resulting logarithm becomes integrable, and thus it is possible to find a constant c satisfying the required Condition 1.

VII. Conclusion

We proved that the rate-splitting schemes for the static 2-user Gaussian IC without CSIT [3], [2] and that with delayed feedback [10], can be extended to the fading case for a class of fading distributions. The proof for optimality to within a constant gap, relies on the sufficient condition, which the fading distribution is assumed to satisfy. We then developed a scheme for symmetric ICs, which can be implemented using point-to-point codes and can approximately achieve the capacity region. An important direction to study will be to see if similar scheme can be extended to general ICs. Also our schemes does not work for bursty IC since it does not satisfy the Condition 1, it would be interesting to study if the schemes can be extended to bursty IC and then to any arbitrary fading distribution.

Appendix A
Proof of achievability for feedback case

We evaluate the term in the first inner bound inequality (16) . The other terms can be similarly evaluated.

$$I\left(U,U_2,X_1;Y_1,\underline{g_1}\right) \stackrel{(a)}{=} I\left(U,U_2,X_1;Y_1|\underline{g_1}\right) \tag{66}$$

$$= h\left(Y_1|\underline{g_1}\right) - h\left(Y_1|\underline{g_1},U,U_2,X_1\right), \tag{67}$$

$$\text{variance}\left(Y_1|\underline{g_1}\right) = \text{variance}\left(g_{11}X_1 + g_{21}X_2 + Z_1|g_{11},g_{21}\right) \tag{68}$$

$$= |g_{11}|^2 + |g_{21}|^2 + g_{11}^* g_{21}\mathbb{E}\left[X_1^* X_2\right] + g_{11}g_{21}^*\mathbb{E}\left[X_1 X_2^*\right] + 1 \tag{69}$$

$$= |g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \operatorname{Re}\left(g_{11}g_{21}^* e^{i\theta}\right) + 1 \tag{70}$$

$$h\left(Y_1|\underline{g_1},U,U_2,X_1\right) = h\left(g_{11}X_1 + g_{21}X_2 + Z_1|\underline{g_1},U,U_2,X_1\right) \tag{71}$$

$$= h\left(g_{21}X_{p2} + Z_1|\underline{g_1}\right) \tag{72}$$

$$= \mathbb{E}\left[\log\left(1 + \lambda_{p2}|g_{21}|^2\right)\right] + \log(2\pi e) \tag{73}$$

$$\stackrel{(b)}{\leq} \mathbb{E}\left[\log\left(1 + \frac{1}{INR_2}|g_{21}|^2\right)\right] + \log(2\pi e) \tag{74}$$

$$\stackrel{(c)}{\leq} \log(2) + \log(2\pi e) \tag{75}$$

$$= 1 + \log(2\pi e), \tag{76}$$

$$\therefore I\left(U,U_2,X_1;Y_1,\underline{g_1}\right) \geq \mathbb{E}\left[\log\left(|g_{11}|^2 + |g_{21}|^2 + 2|\rho|^2 \operatorname{Re}\left(g_{11}g_{21}^* e^{i\theta}\right) + 1\right)\right] - 1 \tag{77}$$

where (a) uses independence, (b) uses the monotonicity of expectation and $\lambda_{pi} \leq \frac{1}{INR_i}$, and (c) follows from Jensen's inequality.

Appendix B
Proof of outer bounds for feedback case

Following the Suh-Tse methods [10], we let $\mathbb{E}\left[X_1X_2^*\right] = \rho$. We have the notation $\underline{g_1} = [g_{11}, g_{21}]$, $\underline{g_2} = [g_{22}, g_{12}]$, $\underline{g} = [g_{11}, g_{21}, g_{22}, g_{12}]$, $S_1 = g_{12}X_1 + Z_2$, and $S_2 = g_{21}X_2 + Z_1$. We let $\mathbb{E}\left[X_1X_2^*\right] = \rho = |\rho|\, e^{i\theta}$. On choosing a uniform distribution of messages we get

$$n(R_1-\epsilon_n) \overset{(a)}{\leq} I\left(W_1;Y_1^n|\underline{g_1^n}\right) \tag{78}$$

$$\overset{(b)}{\leq} \sum \left(h\left(Y_{1i}|\underline{g_{1i}}\right)-h\left(Z_{1i}\right)\right) \tag{79}$$

$$= \sum \left(\mathbb{E}_{\underline{\tilde{g}_{1i}}}\left[h\left(Y_{1i}|\underline{g_{1i}}=\underline{\tilde{g}_{1i}}\right)-h\left(Z_{1i}\right)\right]\right) \tag{80}$$

$$\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}_1}}\left[\sum \left(h\left(Y_{1i}|\underline{g_{1i}}=\underline{\tilde{g}_1}\right)-h\left(Z_{1i}\right)\right)\right] \tag{81}$$

$$\therefore R_1 \leq \mathbb{E}\left[\log\left(|g_{11}|^2+|g_{21}|^2+(\rho^* g_{11}^* g_{21}+\rho g_{11} g_{21}^*)+1\right)\right] \tag{82}$$

where (a) follows from Fano's inequality, (b) follows from the fact that conditioning reduces entropy, and (c) follows from the fact that $\underline{\tilde{g}_{1i}}$ are i.i.d. Now we bound R_1 in a second way as done in [10]:

$$n(R_1-\epsilon_n) \leq I\left(W_1;Y_1^n,\underline{g_1^n}\right) \tag{83}$$

$$\leq I\left(W_1;Y_1^n,\underline{g_1^n},Y_2^n,\underline{g_2^n},W_2\right) \tag{84}$$

$$= I\left(W_1;\underline{g_1^n},\underline{g_2^n},W_2\right)+I\left(W_1;Y_1^n,Y_2^n|\underline{g_1^n},\underline{g_2^n},W_2\right) \tag{85}$$

$$= 0+I\left(W_1;Y_1^n,Y_2^n|\underline{g}^n,W_2\right) \tag{86}$$

$$= h\left(Y_1^n,Y_2^n|\underline{g}^n,W_2\right)-h\left(Y_1^n,Y_2^n|\underline{g}^n,W_1,W_2\right) \tag{87}$$

$$= \sum\left[h\left(Y_{1i},Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1}\right)\right]-\sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right] \tag{88}$$

$$= \sum\left[h\left(Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1}\right)\right]+\sum\left[h\left(Y_{1i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i}\right)\right] - \sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right] \tag{89}$$

$$\overset{(a)}{=} \sum\left[h\left(Y_{2i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i-1},X_2^i\right)\right]+\sum\left[h\left(Y_{1i}|\underline{g}^n,W_2,Y_1^{i-1},Y_2^{i},S_{1i},X_2^i\right)\right] - \sum\left[h\left(Z_{1i}\right)+h\left(Z_{2i}\right)\right] \tag{90}$$

$$\overset{(b)}{\leq} \sum\left[h\left(Y_{2i}|\underline{g_i},X_{2i}\right)-h\left(Z_{2i}\right)\right]+\sum\left[h\left(Y_{1i}|\underline{g_i},S_{1i},X_{2i}\right)-h\left(Z_{1i}\right)\right] \tag{91}$$

$$\overset{(c)}{=} \mathbb{E}_{\underline{\tilde{g}}}\left[\sum\left(h\left(Y_{2i}|X_{2i},\underline{g_i}=\underline{\tilde{g}}\right)-h\left(Z_{2i}\right)\right)\right] + \mathbb{E}_{\underline{\tilde{g}}}\left[\sum\left(h\left(Y_{1i}|S_{1i},X_{2i},\underline{g_i}=\underline{\tilde{g}}\right)-h\left(Z_{1i}\right)\right)\right] \tag{92}$$

$$\therefore R_1 \overset{(d)}{\leq} \mathbb{E}\left[\log\left(1+\left(1-|\rho|^2\right)|g_{12}|^2\right)\right]+\mathbb{E}\left[\log\left(1+\frac{\left(1-|\rho|^2\right)|g_{11}|^2}{1+\left(1-|\rho|^2\right)|g_{12}|^2}\right)\right] \tag{93}$$

where (a) follows from the fact that X_2^i is a function of $\left(W_2, Y_2^{i-1}, \underline{g}^{i-1}\right)$ and S_1^i is a function of (Y_2^i, X_2^i), (b) follows from the fact that conditioning reduces entropy, (c) follows from the fact that $\underline{\tilde{g}_i}$ are i.i.d., and (d) follows from the Suh-Tse results [10]. The other outer bounds can be derived similarly following [10] and making suitable changes to account for fading as we illustrated in the previous two derivations.

APPENDIX C
FADING MATRIX

The calculations are given in equations (94,94).

$$\begin{aligned}
&\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right] \\
&= \mathbb{E}\left[\log\left(\left(|g_{11}(n)|^2 + |g_{21}(n)|^2\left(\frac{|g_{12}(n)|^2+1}{1+INR}\right)+1\right)|K_{\mathbf{Y_1}}(n-1)|\right.\right. \\
&\qquad\left.\left. - \frac{|g_{11}(n-1)|^2|g_{21}(n)|^2|g_{12}(n-1)|^2}{1+INR}|K_{\mathbf{Y_1}}(n-2)|\right)\right] && (94)\\
&\geq \mathbb{E}\left[\log\left((1+INR+SNR)|K_{\mathbf{Y_1}}(n-1)|\right.\right. \\
&\qquad\left.\left. - \frac{INR\,|g_{11}(n-1)|^2|g_{12}(n-1)|^2}{1+INR}|K_{\mathbf{Y_1}}(n-2)|\right)\right] - 3c && (95)
\end{aligned}$$

The first step (94), is by expanding the determinant. We use the Condition 1 thrice in the second step (94). This is justified because the coefficients of $\left\{|g_{11}(n)|^2, |g_{12}(n)|^2, |g_{21}(n)|^2\right\}$ in step (94) are non-negative (due to the fact that all the matrices involved are covariance matrices), and the coefficients themselves are independent of$\left\{|g_{11}(n)|^2, |g_{12}(n)|^2, |g_{21}(n)|^2\right\}$. This procedure can be carried out n times and it follows that:

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\log\left(|K_{\mathbf{Y_1}}(n)|\right)\right] \geq \lim_{n\to\infty}\frac{1}{n}\log\left(\left|\hat{K}_{\mathbf{Y_1}}(n)\right|\right) - 3c \tag{96}$$

where $\hat{K}_{\mathbf{Y_1}}(n)$ is obtained from $K_{\mathbf{Y_1}}(n)$ by replacing $g_{12}(i)$'s,$g_{21}(i)$'s with $\sqrt{INR}$ and $g_{11}(i)$'s with $\sqrt{SNR}$.

APPENDIX D
MATRIX DETERMINANT: ASYMPTOTIC BEHAVIOR

The following recursion easily follows:

$$|A_n| = |a|\,|A_{n-1}| - |b|^2\,|A_{n-2}| \tag{97}$$

with $|A_1| = |a|, |A_2| = |a|^2 - |b|^2$. Also $|A_0|$ can be consistently defined to be 1. The characteristic equation for this recursive relation is given by: $\lambda^2 - |a|\lambda + |b|^2 = 0$ and the characteristic roots are given by:

$$\lambda_1 = \frac{|a| + \sqrt{|a|^2 - 4|b|^2}}{2}, \lambda_2 = \frac{|a| - \sqrt{|a|^2 - 4|b|^2}}{2}. \tag{98}$$

Now the solution of the recursive system is given by:

$$|A_n| = c_1\lambda_1^n + c_2\lambda_2^n \tag{99}$$

with the boundary conditions

$$1 = c_1 + c_2 \tag{100}$$

$$|a| = c_1\lambda_1 + c_2\lambda_2. \tag{101}$$

It can be easily seen that $c_1 > 0$, $\lambda_1 > \lambda_2 > 0$ since $|a|^2 > 4|b|^2$ by assumption of Lemma 9. Now

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log(|A_n|) &= \lim_{n\to\infty} \frac{1}{n} \log(c_1\lambda_1^n + c_2\lambda_2^n) && (102)\\
&= \lim_{n\to\infty} \frac{1}{n} \left(\log(\lambda_1^n) + \log\left(c_1 + c_2\frac{\lambda_2^n}{\lambda_1^n}\right)\right) && (103)\\
&\overset{(a)}{=} \log(\lambda_1) && (104)\\
&= \log\left(\frac{|a| + \sqrt{|a|^2 - 4|b|^2}}{2}\right). && (105)
\end{aligned}$$

The step (a) follows because $1 > \frac{\lambda_2}{\lambda_1} > 0$ and $c_1 > 0$.

Appendix E

Approximate capacity using N phase schemes

We have the following outerbounds from Theorem 3.

$$R_1, R_2 \le \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \tag{106}$$

$$R_1 + R_2 \le \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1 + |g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 2|g_d||g_c| + 1\right)\right] \tag{107}$$

The above outer-bound region is a polytope with the following two non-trivial corner points:

$$\left\{ \begin{array}{lcc} R_1 & = & \mathbb{E}\left[\log\left(|g_d|^2 + |g_c|^2 + 1\right)\right] \\ R_2 & = & \mathbb{E}\left[\log\left(1 + \frac{|g_d|^2}{1+|g_c|^2}\right)\right] + \mathbb{E}\left[\log\left(1 + \frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right] \end{array} \right\}$$

$$\left\{ \begin{array}{lcl} R_1 & = & \mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]+\mathbb{E}\left[\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right] \\ R_2 & = & \mathbb{E}\left[\log\left(|g_d|^2+|g_c|^2+1\right)\right] \end{array} \right\}$$

We can achieve these rate points within $2+2c$ bits for each user using the n-phase schemes since

$$(R_1,R_2)=\left(\log\left(1+SNR+INR\right)-2-2c,\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right]\right) \tag{108}$$

$$(R_1,R_2)=\left(\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right],\log\left(1+SNR+INR\right)-2-2c\right) \tag{109}$$

are achievable and since using Jensen's inequality

$$\mathbb{E}\left[\log\left(|g_d|^2+|g_c|^2+1\right)\right]\leq\log\left(1+SNR+INR\right). \tag{110}$$

The only important point left to verify is in the following claim

Claim 17. $\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]+\mathbb{E}\left[\log\left(1+\frac{2|g_d||g_c|}{1+|g_d|^2+|g_c|^2}\right)\right]-\mathbb{E}\left[\log^+\left[\frac{|g_d|^2}{1+INR}\right]\right]\leq 2+c$

Proof: We have $\frac{2|g_d||g_c|}{|g_d|^2+|g_c|^2}\leq 1$ due to AM-GM inequality. Hence

$$\mathbb{E}\left[\log\left(1+\frac{2\left|g_d\right|\left|g_c\right|}{1+|g_d|^2+|g_c|^2}\right)\right]\leq 1. \tag{111}$$

Also

$$\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+|g_c|^2}\right)\right]\leq\mathbb{E}\left[\log\left(1+\frac{|g_d|^2}{1+INR}\right)\right]+c \tag{112}$$

using Condition 1. Hence it only remains to show $\log\left(1+\frac{|g_d|^2}{1+INR}\right)-\log^+\left[\frac{|g_d|^2}{1+INR}\right]\leq 1$ to complete the proof.

Now if $\log^+\left[\frac{|g_d|^2}{1+INR}\right]=0$ then $\frac{|g_d|^2}{1+INR}\leq 1$ and hence $\log\left(1+\frac{|g_d|^2}{1+INR}\right)\leq\log\left(2\right)=1$. If $\log^+\left[\frac{|g_d|^2}{1+INR}\right]>0$ then $\frac{|g_d|^2}{1+INR}>1$ and hence again

$$\log\left(1+\frac{|g_d|^2}{1+INR}\right)-\log^+\left[\frac{|g_d|^2}{1+INR}\right]=\log\left(1+\frac{1+INR}{|g_d|^2}\right)<1. \tag{113}$$

■

APPENDIX F
PROOF OF LEMMA 16

We have $F(w) \leq aw^b$ for $w \in [0, \epsilon]$ where $a \geq 0, b > 0, 1 \geq \epsilon > 0$. Now using integration by parts we get

$$\begin{aligned}
\mathbb{E}[\ln(W)] &\geq \int_0^1 f(w) \ln(w) \\
&= \int_0^\epsilon f(w) \ln(w) + \int_\epsilon^1 f(w) \ln(w) \\
&= [F(w) \ln(w)]_0^\epsilon - \int_0^\epsilon F(w) \frac{1}{w} + \int_\epsilon^1 f(w) \ln(w) \\
&\geq \left[aw^b \ln(w)\right]_0^\epsilon - \int_0^\epsilon aw^b \frac{1}{w} \\
&\geq a\epsilon^b \ln(\epsilon) - \frac{a\epsilon^b}{b} + \ln(\epsilon).
\end{aligned}$$

Note that $\ln(w)$ is negative in the range $[0, 1)$, thus we get the desired inequalities in the previous steps.

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